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Computational Aspects of Prediction Markets

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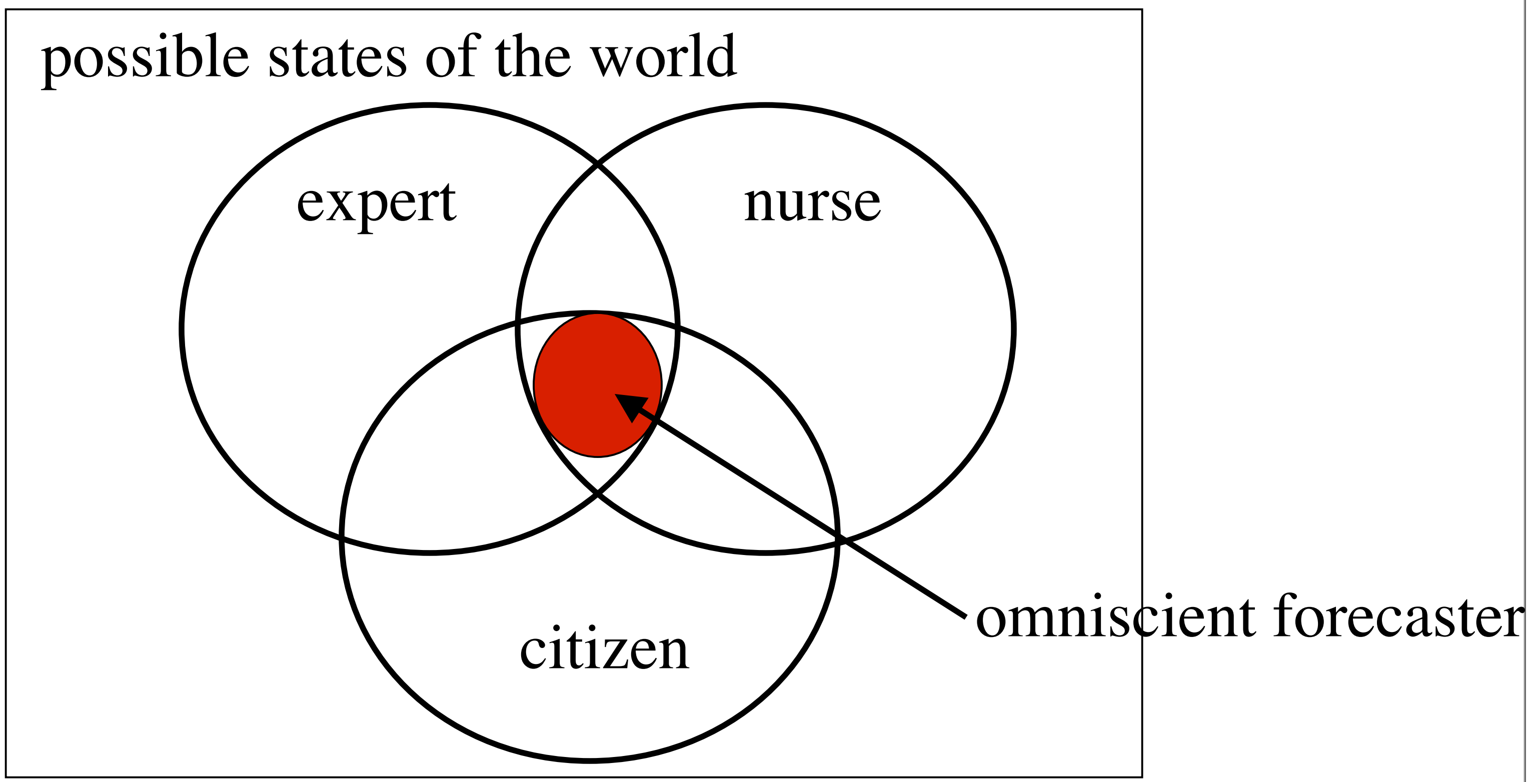
Mech Design for Prediction

- Q: Will there be a bird flu outbreak in the UK in 2007?
- A: Uncertain. Evidence distributed: health experts, nurses, public
- Goal: Obtain a forecast as good as omniscient center with access to all evidence from all sources

Mech Design for Prediction



A Prediction Market

- **Take a random variable, e.g.**

Bird Flu Outbreak UK 2007?
(Y/N)

- **Turn it into a financial instrument payoff = realized value of variable**

I am entitled to:

$1 if	Bird Flu UK ’07	$0 if	Bird Flu UK ’07

Tradesports.com
START TRADING

http://tradesports.com

Contract	BQty	Bid	Offer	AQty	Last	Vol	Chge
Trade OSAMA.CAPTURE.MAR07	5	6.3	8.5	5	8.9	730	+0.9
Trade OSAMA.CAPTURE.JUN07	23	11.1	13.2	5	13.3	210	0
Trade OSAMA.CAPTURE.SEP07	10	15.3	18.1	1	16.6	174	0
Trade OSAMA.CAPTURE.DEC07	1	20.0	22.0	1	21.5	640	0
Trade OSAMA.CAPTURE.DEC06	Expired at 0.0					11.1k	-20.0

Contract	BQty	Bid	Offer	AQty	Last	Vol	Chge
Trade BIRDFLU.USA.31MAR07	1	7.0	13.0	7	9.0	781	0
Trade BIRDFLU.USA.31DEC06	Expired at 0.0					3627	-65.0

Contract	BQty	Bid	Offer	AQty	Last	Vol	Chge
Trade NFL.CHARGERS	102	26.5	26.6	8	26.5	35.9k	-0.3
Trade NFL.BEARS	4	14.2	14.4	3	14.5	37.3k	-0.4
Trade NFL.COLTS	100	8.0	8.7	3	8.8	27.1k	+1.0
Trade NFL.RAVENS	977	16.4	16.5	2	16.5	35.9k	+1.1
Trade NFL.SAINTS	16	9.7	9.8	12	9.7	35.0k	-0.5

Mech Design for Prediction

- Standard Properties
  - ~~Efficiency~~
  - Inidiv. rationality
  - ~~Budget balance~~
  - ~~Revenue~~
  - Comp. complexity
- Equilibrium
  - General, Nash, ...

- PM Properties
  - #1: Info aggregation
  - Expressiveness
  - Liquidity
  - Bounded budget
  - Indiv. rationality
  - Comp. complexity
- Equilibrium
  - Rational expectations

Competes with: experts, scoring rules, opinion pools, ML/stats, polls, Delphi

Outline

- **Some computational aspects of PMs**
  - **Combinatorics**
    - Betting on permutations
    - Betting on Boolean expressions
  - **Automated market makers**
    - Hanson's market scoring rules
    - Dynamic parimutuel market
  - **(Computational model of a market)**

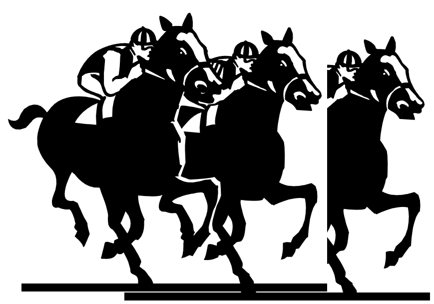
Predicting Permutations

- Predict the ordering of a set of statistics
  - Horse race finishing times
  - Daily stock price changes
  - NFL Football quarterback passing yards
  - Any ordinal prediction
- Chen, Fortnow, Nikolova, Pennock, EC’07

Market Combinatorics

Permutations

- A > B > C .1
- A > C > B .2
- B > A > C .1
- B > C > A .3
- C > A > B .1
- C > B > A .2



Market Combinatorics

Permutations

- D > A > B > C .01
- D > A > C > B .02
- D > B > A > C .01
- A > D > B > C .01
- A > D > C > B .02
- B > D > A > C .05
- A > B > D > C .01
- A > C > D > B .2
- B > A > D > C .01
- A > B > C > D .01
- A > C > B > D .02
- B > A > C > D .01
- D > B > C > A .05
- D > C > A > B .1
- D > C > B > A .2
- B > D > C > A .03
- C > D > A > B .1
- C > D > B > A .02
- B > C > D > A .03
- C > A > D > B .01
- C > B > D > A .02
- [illegible] > [illegible] > D > A .03
- [illegible] > [illegible] > D > B .01
- C > B > D > A .02

Bidding Languages

- Traders want to bet on *properties* of orderings, not explicitly on orderings: more natural, more feasible
  - A will win ; A will “show”
  - A will finish in [4-7] ; {A,C,E} will finish in top 10
  - A will beat B ; {A,D} will both beat {B,C}
- *Buy 6 units of “$1 if A>B” at price $0.4*
- Supported to a limited extent at racetrack today, but *each in different betting pools*
- Want centralized auctioneer to improve liquidity & information aggregation

Auctioneer Problem

- **Auctioneer's goal:**
  Accept orders with non-zero worst-case loss (auctioneer never loses money)

  The Matching Problem

- **Formulated as LP**

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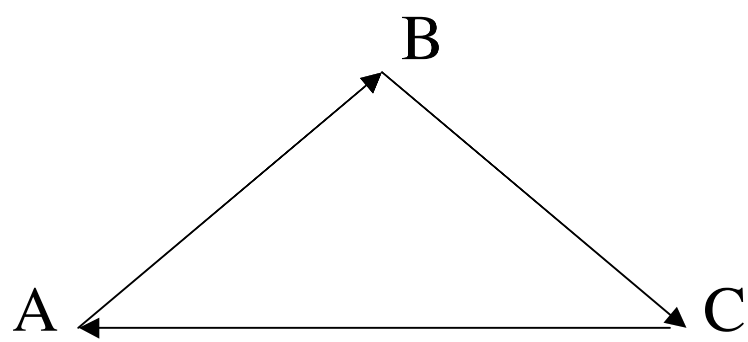
Example

- **A three-way match**
  - **Buy 1 of "$1 if A>B" for 0.7**
  - **Buy 1 of "$1 if B>C" for 0.7**
  - **Buy 1 of "$1 if C>A" for 0.7**



Pair Betting

- All bets are of the form “A will beat B”
- Cycle with sum of prices > k-1 ==> Match (Find best cycle: Polytime)
- Match =/=> Cycle with sum of prices > k-1
- Theorem: The Matching Problem for Pair Betting is NP-hard (reduce from min feedback arc set)

Subset Betting

- All bets are of the form
  - "A will finish in positions 3-7", or
  - "A will finish in positions 1,3, or 10", or
  - "A, D, or F will finish in position 2"

- Theorem: The Matching Problem for Subset Betting is polytime (LP + maximum matching separation oracle)

Market Combinatorics

Boolean

I am entitled to: \$1 if A1&A2&…&An

I am entitled to: \$1 if A1&A2&…&$\overline{\text{An}}$

I am entitled to: \$1 if $\overline{\text{A1}}$&A2&…&An

I am entitled to: \$1 if $\overline{\text{A1}}$&A2&…&$\overline{\text{An}}$

I am entitled to: \$1 if A1&$\overline{\text{A2}}$&…&An

I am entitled to: \$1 if A1&$\overline{\text{A2}}$&…&$\overline{\text{An}}$

I am entitled to: \$1 if $\overline{\text{A1}}$&$\overline{\text{A2}}$&…&An

I am entitled to: \$1 if $\overline{\text{A1}}$&$\overline{\text{A2}}$&…&$\overline{\text{An}}$

- **Betting on complete conjunctions is both unnatural and infeasible**

Market Combinatorics

Boolean

- A bidding language: write your own security
  - I am entitled to: $1 if Boolean_fn | Boolean_fn
- For example
  - I am entitled to: $\$1 \text{ if } A1 \mid \overline{A2}$
  - I am entitled to: $\$1 \text{ if } A1\&\overline{A7}$
  - I am entitled to: $\$1 \text{ if } (A1\&\overline{A7})||A13 \mid (A2||\overline{A5})\&A9$
- Offer to buy/sell q units of it at price p
- Let everyone else do the same
- Auctioneer must decide who trades with whom at what price… How? (next)
- More concise/expressive; more natural

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The Matching Problem

- **There are many possible matching rules for the auctioneer**
- **A natural one: maximize trade subject to no-risk constraint**
- **Example:**
  - **buy 1 of** \$1 if A1 **for \$0.40**
  - **sell 1 of** \$1 if A1&A2 **for \$0.10**
  - **sell 1 of** \$1 if A1& $\overline{A2}$ **for \$0.20**
- **No matter what happens, auctioneer cannot lose money**

trader gets \$\$ in state:

$A1A2$	$A1\overline{A2}$	$\overline{A1}A2$	$\overline{A1A2}$
0.60	0.60	-0.40	-0.40
-0.90	0.10	0.10	0.10
0.20	-0.80	0.20	0.20
-0.10	-0.10	-0.10	-0.10

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Market Combinatorics

Boolean

Prediction Markets for 2006 US Senate Races

Contract	BQty	Bid	Offer	AQty	Last	Vol	Chge
Trade ALABAMA.DEM	100	5.0	15.0	100	8.0	0	0
Trade ALABAMA.REP	2	85.1	95.0	100	90.0	1	0
Trade ALABAMA.FIELD	0	-	5.0	100	2.5	0	0
Trade ALASKA.DEM	100	10.0	20.0	100	14.0	0	0
Trade ALASKA.REP	1	80.1	90.0	100	85.0	0	0
Trade ALASKA.FIELD	0	-	5.0	100	2.5	0	0
Trade ARIZONA.DEM	100	27.0	35.0	100	28.0	10	0
Trade ARIZONA.REP	100	65.0	75.0	100	70.0	10	0
Trade ARIZONA.FIELD	0	-	5.0	100	2.5	0	0
Trade ARKANSAS.DEM	100	25.0	30.0	71	26.0	30	0
Trade ARKANSAS.REP	100	70.0	80.0	100	75.0	0	0
Trade ARKANSAS.FIELD	0	-	5.0	100	2.5	0	0

Predicted Probabilities of Senate Elections based on Market Data from Tradesports.com

Expected Republican 50.78 Democrat 47.25 Others 1.98
Leaning Democrat 49 Republican 49 Others 2

GOP Senate Control 69.0%
GOP House Control 20.0%

Fortnow; Kilian; Pennock; Wellman

Complexity Results

- **Divisible orders: will accept any q* ≤ q**
- **Indivisible: will accept all or nothing**

# events	divisible	indivisible
O(log n)	polynomial	NP-complete
O(n)	co-NP-complete	Σ_2^p complete

LP

reduction from X3C

reduction from SAT

reduction from T∃∀BF

- **Natural algorithms**
  - **divisible: linear programming**
  - **indivisible: integer programming; logical reduction?**

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[Thanks: Yiling Chen]

Automated Market Makers

- A market maker (a.k.a. bookmaker) is a firm or person who is almost always willing to accept both buy and sell orders at some prices
- Why an institutional market maker? Liquidity!
  - Without market makers, the more expressive the betting mechanism is the less liquid the market is (few exact matches)
  - Illiquidity discourages trading: Chicken and egg
  - Subsidizes information gathering and aggregation: Circumvents no-trade theorems
- Market makers, unlike auctioneers, bear risk. Thus, we desire mechanisms that can bound the loss of market makers
  - Market scoring rules [Hanson 2002, 2003, 2006]
  - Dynamic pari-mutuel market [Pennock 2004]

[Thanks: Yiling Chen]

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Automated Market Makers

- n disjoint and exhaustive outcomes
- Market maker maintain vector Q of outstanding shares
- Market maker maintains a cost function C(Q) recording total amount spent by traders
- To buy ΔQ shares trader pays C(Q+ ΔQ) – C(Q) to the market maker; Negative “payment” = receive money
- Instantaneous price functions are $p_i(Q) = \frac{\partial C(Q)}{\partial q_i}$
- At the beginning of the market, the market maker sets the initial Q^0, hence subsidizes the market with $C(Q^0)$.
- At the end of the market, $C(Q^f)$ is the total money collected in the market. It is the maximum amount that the MM will pay out.

[Thanks: Yiling Chen]

Hanson's Market Maker I

Logarithmic Market Scoring Rule

- ***n*** **mutually exclusive outcomes**
- **Shares pay $1 if and only if outcome occurs**
- **Cost Function**

$$C(Q) = b \times \log\left(\sum_{i=1}^{n} e^{\frac{q_i}{b}}\right)$$

- **Price Function**

$$p_i(Q) = \frac{e^{\frac{q_i}{b}}}{\sum_{j=1}^{n} e^{\frac{q_j}{b}}}$$

[Thanks: Yiling Chen]

Hanson's Market Maker II

Quadratic Market Scoring Rule

- **We can also choose different cost and price functions**
- **Cost Function**

$$C(Q) = \frac{\sum_{i=1}^{n} q_i}{n} + \frac{\sum_{i=1}^{n} q_i^{\,2}}{4b} + \frac{(\sum_{i=1}^{n} q_i)^2}{4b} - \frac{b}{n}$$

- **Price Function**

$$p_i(Q) = \frac{1}{n} + \frac{q_i}{2b} - \frac{\sum_{j=1}^{n} q_j}{2nb}$$

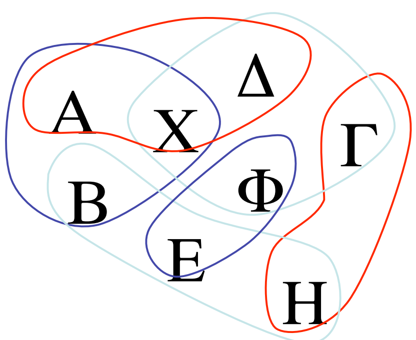
Log Market Scoring Rule

- Market maker's loss is bounded by b * ln(n)
- Higher b $\Rightarrow$more risk, more "liquidity"
- Level of liquidity (b) never changes as wagers are made
  - Could charge transaction fee, put back into b (Todd Proebsting)
- Much more to MSR: sequential shared scoring rule, combinatorial MM "for free", ... see Hanson 2002, 2003, 2006

[Source: Hanson, 2002]

Computational Issues

- Straightforward approach requires exponential space for prices, holdings, portfolios
- Could represent probabilities using a Bayes net or other compact representation; changes must keep distribution in the same representational class
- Could use multiple overlapping patrons, each with bounded loss. Limited arbitrage could be obtained by smart traders exploiting inconsistencies between patrons



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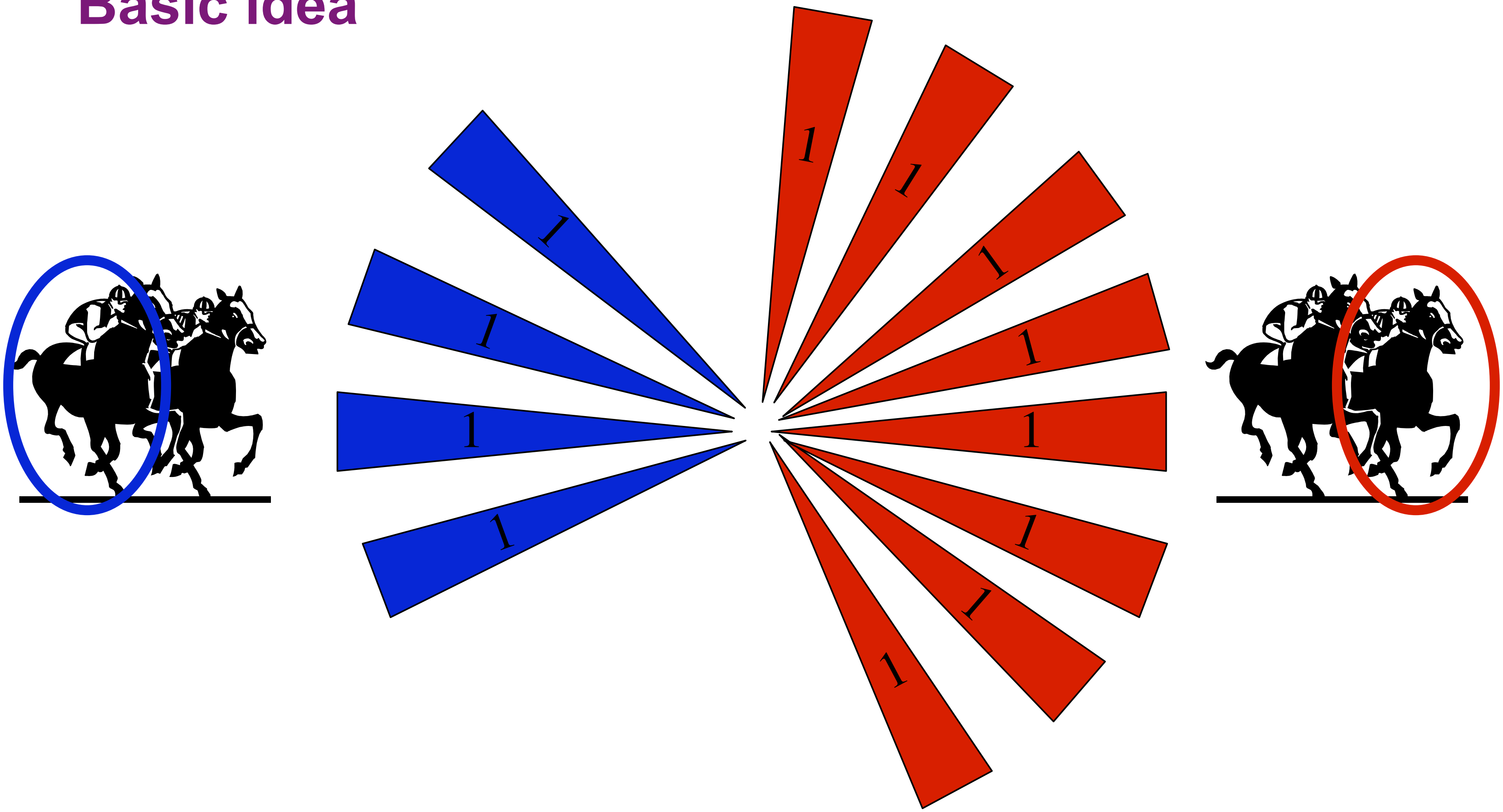
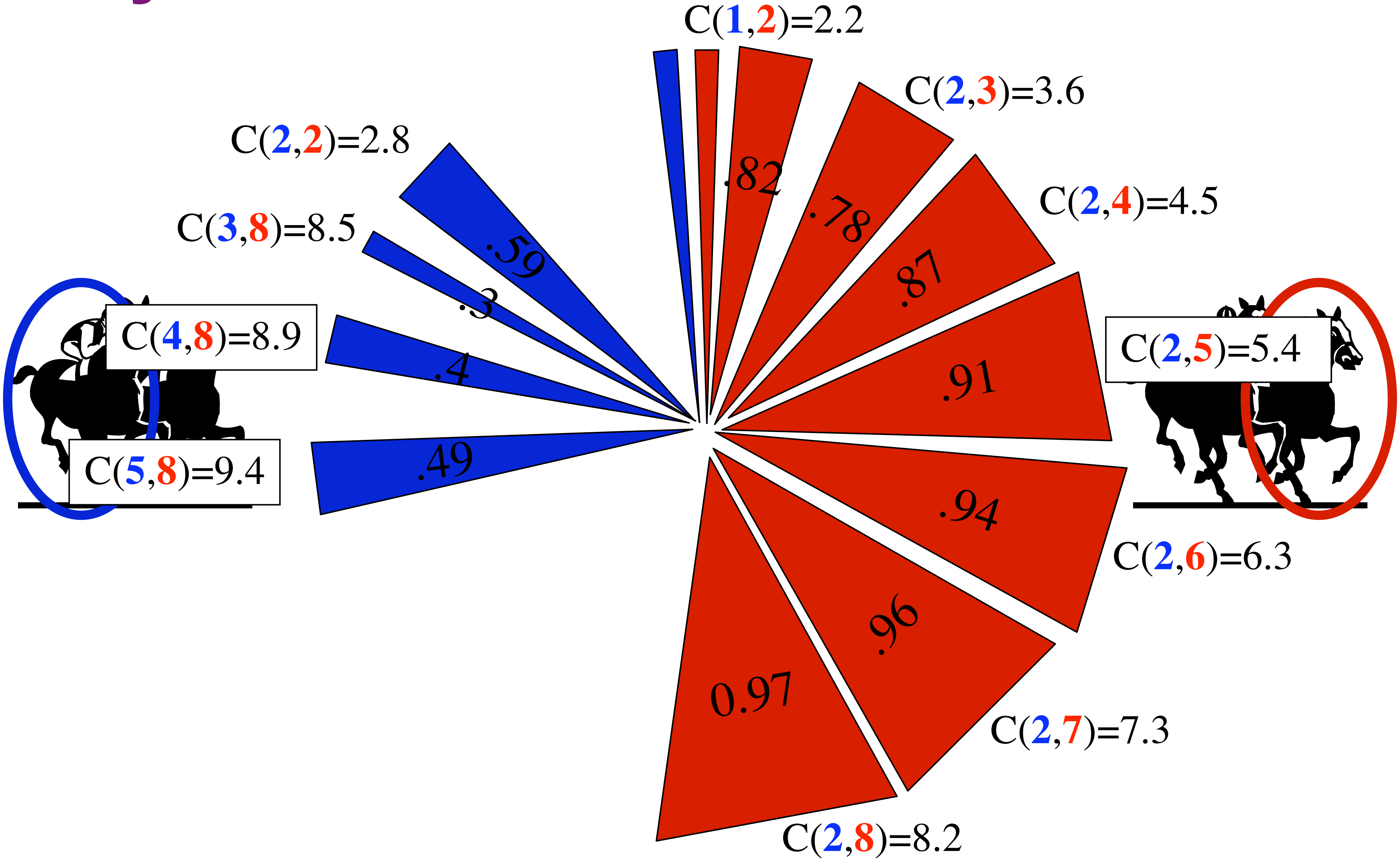
Pari-Mutuel Market

Basic idea



Dynamic Parimutuel Market



Share-ratio price function

- **One can view DPM as a market maker**
- **Cost Function:**

$$C(Q) = \sqrt{\sum_{i=1}^{n} q_i^{\ 2}}$$

- **Price Function:**

$$p_i(Q) = \frac{q_i}{\sqrt{\sum_{j=1}^{n} q_j^{\ 2}}}$$

- **Properties**
  - **No arbitrage**
  - **$price_i / price_j = q_i / q_j$**
  - **$price_i < \$1$**
  - **payoff if right = $C(Q_{final})/q_o > \$1$**

Open Questions
Combinatorial Betting

- **Usual hunt: Are there natural, useful, expressive bidding languages (for permutations, Boolean, other) that admit polynomial time matching?**
- **Are there good heuristic matching algorithms (think WalkSAT for matching); logical reduction?**
- **How can we divide the surplus?**
- **What is the complexity of incremental matching?**

Open Questions
Automated Market Makers

- For every bidding language with polytime matching, does there exist a polytime MSR market maker?
- The automated MM algorithms are online algorithms: Are there other online MM algorithms that trade more for same loss bound?