Duality

Tuesday, November 10, 2015
9:23 AM

- Let's return to the plain idli - rawa idli example

$$\max \quad P + 3R$$
$$\text{s.t.} \quad P \leq 100$$
$$R \leq 150$$
$$P + R \leq 200$$
$$P, R \geq 0$$

- We said that $P^* = 50$, $R^* = 150$ is the optimum; with obj value 500 is the optimum solution. Can we prove this?
- Note that:

$$OPT = \max_{P,R} (P + 3R) = \max_{P,R} ((P+R) + 2R)$$
$$\leq 200 + 2 \cdot 150 = 500$$

So, (P^*, R^*) must be optimal!

- $$y_1 (P) \leq 100 y_1$$ Assuming $y_1, y_2, y_3 \geq 0$.
$$y_2 (R) \leq 150 y_2$$
$$y_3 (P+R) \leq 200 y_3$$
$$\overline{(y_1 + y_3) P + (y_2 + y_3) R \leq 100 y_1 + 150 y_2 + 200 y_3}$$

- Then, $P + 2Q \leq 100 y_1 + 150 y_2 + 200 y_3$

if
$$1 \leq y_1 + y_3$$
$$2 \leq y_2 + y_3$$
$$y_1, y_2 \geq 0$$

- Dual program: $\min \quad 100 y_1 + 150 y_2 + 200 y_3$
$$\text{s.t.} \quad y_1 + y_3 \geq 1$$

$$y_2 + y_3 \geq 2$$

$$y_1, y_2 \geq 0$$

- Any feasible solution for dual gives an upper bound on primal optimal

- Generally:

$$\text{Primal}\begin{cases} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{cases} \longleftrightarrow \left.\begin{matrix} \min & b^T y \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{matrix}\right\}\text{Dual}$$

- ~~Weak duality~~: If x is feasible for primal and y is feasible for dual, then: $c^T x \leq b^T y$

Pf: $$\sum_{i=1}^{n} c_i x_i \leq \sum_{i=1} \left(\sum_{j=1} A_{ij} y_j \right) x_i$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} A_{ij} x_i \right) y_j$$

$$\leq \sum_{j=1}^{m} b_j y_j$$

- If dual (primal) is feasible, then primal (dual) is bounded.

Primal \ Dual	Infeasible	Unbounded	Neither
Infeasible			
Unbounded			
Neither			

- ~~Strong Duality~~: If both primal and dual are bounded & feasible then $OPT_{primal} = OPT_{dual}$

~~Examples~~:

1) Max Matching vs. Min Vertex Cover

$$\max \sum_{e \in E} x_e$$

$$\text{s.t.} \sum_{e:\, e \text{ incident to } v} x_e \leq 1 \quad \forall v \in L$$

$$\sum_{e:\, e \text{ incident to } v} x_e \leq 1 \quad \forall v \in R$$

$$x_e \geq 0$$

~~Dual~~: $$\min \sum_{v \in L \cup R} y_v$$

$$\text{s.t.} \quad y_u + y_v \geq 1 \quad \forall (u,v) \in E$$

$$y_v \geq 0 \quad \forall v \in L \cup R$$

Vertex Cover LP!

2) Set Cover vs. Packing

Instance: Sets $S_1, \ldots, S_m$,

Elements $e_1, \ldots, e_n$

Output: Collection of sets covering all elements

LP: $$\min \sum x_S c_S$$

$$\text{s.t.} \sum_{S:\, S \ni e} x_S \geq 1 \quad \forall e$$

$$x_S \geq 0 \quad \forall$$

Dual:

$$\max \sum y_e$$

$$\text{s.t.} \sum_{e \in S} y_e \leq C_S \quad \forall S$$

$$y_e \geq 0$$

Packing problem!

3) Max Flow vs. Min Cut

$$\max \; v$$

$$\text{s.t.} \begin{bmatrix} \text{Incidence matrix} \end{bmatrix} \begin{bmatrix} f \end{bmatrix} = \begin{array}{c} \begin{bmatrix} v \\ -v \end{bmatrix} \end{array} \begin{array}{l} \leftarrow s \\ \leftarrow t \end{array}$$

$$f \leq c$$

$$f \geq 0$$

Dual: Variable x_v for every vertex and y_e for every edge

$$\min \sum_e y_e c_e$$

$$\text{s.t.} \quad x_u - x_v + y_{(u,v)} \geq 0$$

$$-x_s + x_t \geq 1$$

$$y_e \geq 0$$

Given a cut (A,B), $y_{(u,v)} = 1$ if $u \in A, v \in B$
$= 0$ o.w.

$$x_u = 0 \text{ if } u \in A$$
$$\quad = 1 \text{ if } u \in B$$

Complementary slackness : x, y feasible for primal, dual resp.
x and y are optimal for primal and dual respectively iff

$$x_i > 0 \Rightarrow \sum_{j=1}^{m} A_{ij} y_j = c_i$$

$$\text{and } y_j > 0 \Rightarrow \sum_{i=1}^{n} A_{ij} x_i = b_j$$

Pf : ($\Rightarrow$) Use strong duality and equality in proof of weak duality thm

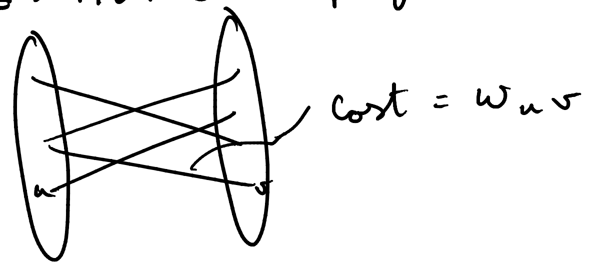
($\Leftarrow$) $\sum x_i c_i = \sum A_{ij} x_i y_j = \sum b_j y_j$

x and y optimal.

Primal dual

Start with feasible dual and keep maintaining feasibility while increasing dual value. Keep going until complementary slackness guarantees primal solution is optimal.

~~Example~~ : Min cost perfect matching in bipartite graphs



Primal : $\min \sum_e w_e x_e$

$$\text{s.t.} \sum_{e \ni v} x_e = 1 \ \forall v \in L$$

$$\sum_{e \ni v} x_e = 1 \ \forall v \in R$$

$$x_e \geq 0$$

Dual : $\max \sum_{u \in L} y_u + \sum_{v \in R} z_v$

$u \in L \qquad v \in R$

$$x_e \geq 0$$

$$\text{s.t.} \quad y_u + z_v \leq w_{uv} \quad \forall (u,v) \in E$$

~~Complementary slackness~~: Suppose x, y are feasible sols to primal & dual resp. Then, x and y are optimal if

$$\cancel{x_{uv} > 0 \Rightarrow y_u + z_v = w_{uv}}$$

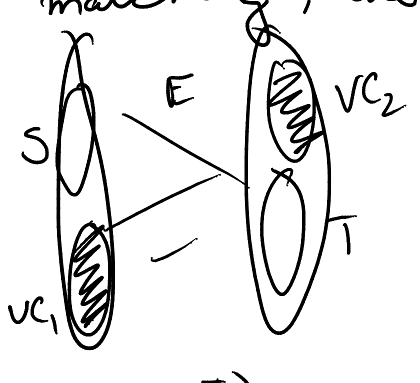
$$y_u \neq 0 \Rightarrow \sum_{e \ni u} x_u = 1$$

$$y_v \neq 0 \Rightarrow \sum_{e \ni v} x_v = 1$$

~~Idea~~: Want to get a feasible solution supported on those e's for $y_u + z_v = w_e$.

$\min w_e$

- Start with $x = 0$, $y_u = 0$, $z^v = \min_{e \ni v} w_e$
- Let $E = \{e : w_e - y_u - z_v = 0\}$.
- Run max cardinality matching on E to get M. If M is a perfect matching, then done!
- Otherwise:

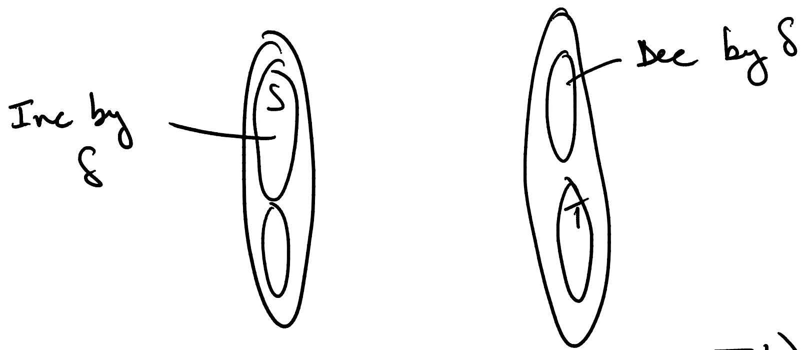


Say min vertex cover is $VC_1 \cup VC_2$.

No edges in E between S and $\bar{T}$!

- For all $(u,v) \in \bar{E} \cap (S \times T)$, $y_u + z_v < w_{uv}$.

$$\delta = \min_{\substack{u \in S \\ v \in T}} (w_{uv} - y_u - z_v)$$



Dual stays feasible!

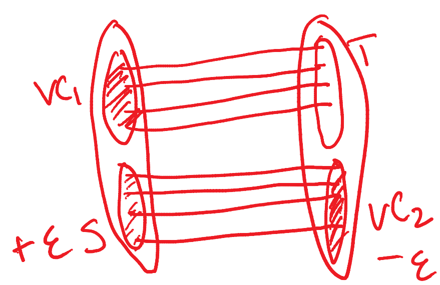
- Dual increases by $\delta(|S| - |R \setminus T|)$
  $= \delta(n - |VC|) > 0$
- The edge set E keeps increasing

Set Cover: Recall dual

$$\max \sum y_e$$
$$\text{s.t.} \sum_{e \in S} y_e \le c_S$$
$$y_e \ge 0$$

Primal-dual approx:

$y_e = 0$
Find uncovered e. Keep raising y_e until some set S becomes tight. Put S in set cover.



Between S and T, no tight edges.

Between VC_1 and VC_2, no edges in the matching.