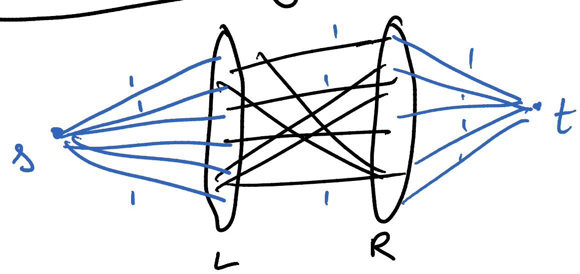
LP's (part ii)

Sunday, November 08, 2015
7:21 AM

- Recall Max Flow-Min Cut Theorem: The max flow in a flow network G is also its min cut. Moreover, the min cut is formed by all the vertices reachable from the source in the residual graph.
- Very useful result. Let's see some applications.

<u>Maximum matching in bipartite graphs</u>



$|L| = |R| = n$

Add s, t as sink and source

All edges unit capacity

<u>Claim 1</u>: $\text{Max Flow}(G) \geq \text{Max Matching}(G)$

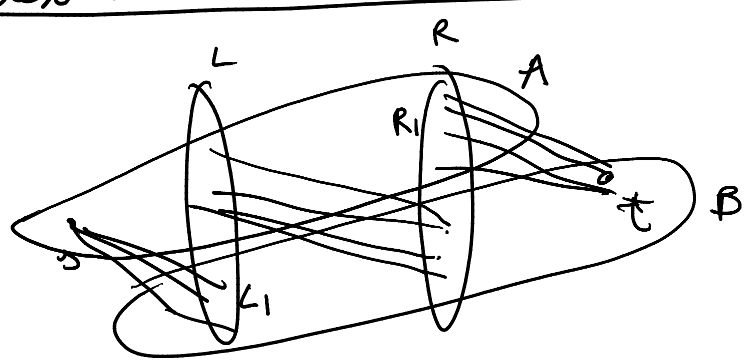
<u>Pf</u>: Any matching gives a valid flow

<u>Claim 2</u>: $\text{Max Flow}(G) \leq \text{Max Matching}(G)$

<u>Pf</u>: FF returns a 0-1 flow, which corresponds to a matching!

<u>Note</u>: This doesn't extend to max wt matchings. We'll see later today how we can do this.

<u>Vertex cover in bipartite graphs</u>



Suppose FF on above graph G produces cut (A, B) with $s \in A, t \in B$.

$L_1 = L \cap B$

$R_1 = R \cap A$

$X = \{v \in R \cap B : \exists u \in L \cap A, (u,v) \in E\}$

Algorithm returns $L_1 \cup R_1 \cup X$ as vertex cover VC.

Claim: VC is a vertex cover

Pf: Clear

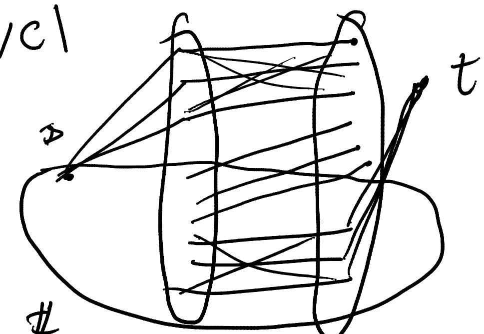
Claim: VC is the minimum vertex cover

Pf: MinCut $\geq |VC|$.

$\Rightarrow$ MaxFlow $\geq |VC|$

$\Rightarrow$ Matching $\geq |VC|$

$\Rightarrow$ Min VC $\geq$ Matching $\geq |VC|$

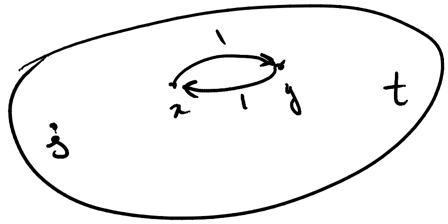


Edge-disjoint paths

Menger's theorem: For any undirected graph G and vertices s, t, the min # of edges required to separate s and t are the # of edge disjoint paths between s and t.

Pf: Clear, that # of edges to separate $\geq$ # of edp.

For other direction:



Unit capacity on every edge

Max flow $\Rightarrow$ edp

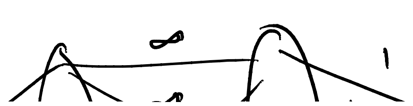
Min cut $\Rightarrow$ # of separating edges

Hall's Theorem

A bipartite graph G has a perfect matching iff

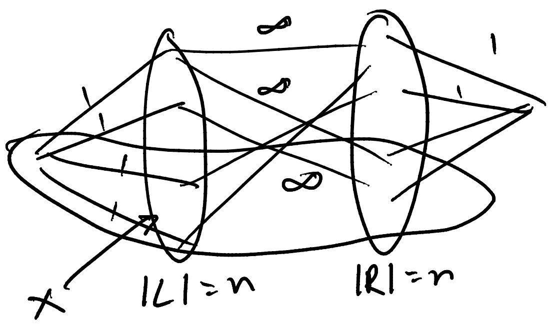
$\forall X \subseteq L$, $|N(X)| \geq |X|$. ($N(X)$ denotes nbrs of X).

Pf:



Min Cut $\leq n$

Pf :



Min Cut $\leq n$

Take any finite value s-t cut and note its value $\geq n$.

$\Rightarrow$ Min Cut $= n$

$\Rightarrow$ Max Flow $= n$

$\Rightarrow$ Perfect matching

Return to LP's

Multi Commodity flow:

$G = (V, E)$, $\quad c : E \to \mathbb{R}^{\geq 0}$

Also, k commodities specified by (s_i, t_i, d_i)

s_i = source

t_i = sink

d_i = demand

$$\min \; 0$$

$$\text{s.t.} \quad \sum_{i=1}^{k} f_i(u,v) \leq c(u,v) \qquad \forall\, (u,v) \in E$$

$$\sum_{u:(u,v)\in E} f_i(u,v) - \sum_{w:(v,w)\in E} f_i(v,w) = \begin{cases} 0 & \forall i, \forall v \notin \{s_i, t_i\} \\ d_i & v \in \{s_i, t_i\} \end{cases}$$

$$f_i(u,v) \geq 0$$

Optimal strategies in games

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

If row player always plays R, col player can always play P.

... for any pure strategy

R	0		
P	1	0	-1
S	-1	1	0

col player can always play 1.

Similarly for any pure strategy

Mix it up: Row player has a prob. distribution over his moves and similarly column player.

Row strategy: (x_1, x_2, x_3)

Col strategy: (y_1, y_2, y_3)

Expected payoff is $\sum G_{ij}\, x_i\, y_j$

$$= \sum_j y_j \sum_i G_{ij}\, x_i$$

Col can just choose the j that minimizes $\sum_i G_{ij}\, x_i$.

So, row wants to maximize $\min_j \left(\sum_i G_{ij}\, x_i\right)$

$$= \max_{x_1, x_2, x_3} \left(x_2 - x_3,\ -x_1 + x_3,\ x_1 - x_2 \right)$$

$$\begin{aligned} \max\quad & z \\ \text{s.t.}\quad & z \le x_2 - x_3 \\ & z \le -x_1 + x_3 \\ & z \le x_1 - x_2 \end{aligned}$$

Solution: $x_1 = x_2 = x_3 = \frac{1}{3}$

Can also check what happens if Col announces strategy first, then Row. Same value! Result of duality, as we explain in next lecture.

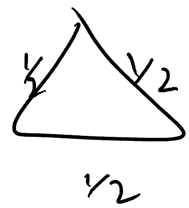
Maximum Wt matching

Consider LP: $\max \sum w_e x_e$

$$\text{s.t.} \quad \sum_{e \ni v} x_e \le 1 \quad \forall v$$

$$x_e \ge 0 \quad \forall e$$

Triangle:



But integral for bipartite graphs!

Pf: $H = \{e : x_e \notin \{0,1\}\}$.

Case 1: H has a cycle. Must be even. Consider alternating $\pm \varepsilon$ on edges.

Case 2: Take longest path in H. Other edges incident to first and last vertices must have wt 0. Again alternate $\pm \varepsilon$ on edges.