## Lecture 7

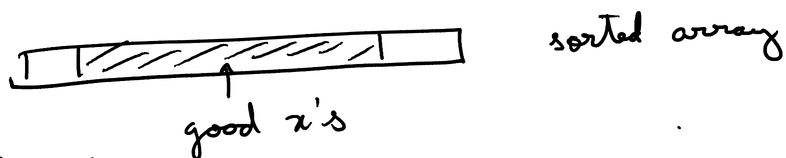
Tuesday, September 01, 2015
2:00 AM

### Skip lists continued

- Recall skip lists
- Showed that $\mathbb{E}[\text{height}] = O(\log n)$
- In search viewed backwards, we go up w.p. $1/2$ and left w.p. $1/2$. So, twice of expected height for searching
- Discuss insertions and deletions

### Randomized selection

- Choose pivot $x$ at random! We expect rank$(x)$ to be close to $n/2$.
- Say $x$ is good for array A if it reduces input size by $0.9$.




- $\Pr[x \text{ is good}] = \frac{4}{5}$
- $\Pr[\text{input size is} > \frac{9}{10} n \text{ even after } k \text{ pivot choices}]$
  $\leq \Pr[\text{no good pivots chosen } k \text{ times in a row}]$
  $= (1/5)^k$
- Let $T_\ell$ be number of recursive calls when input size is between $(9/10)^\ell \cdot n$ and $(9/10)^{\ell+1} \cdot n$.
- $T = \sum_{\ell=0}^{\log_{10/9} n} O(0.9^\ell n) \cdot T_\ell$
- $\mathbb{E}[T] = \sum_{\ell=0}^{\log n} O(0.9^\ell n) \cdot \mathbb{E}[T_\ell]$

- $\mathbb{E}[T] = \sum_{k=0}^{\lg n} O(0.9^{\ell} n) \cdot \mathbb{E}[T_\ell]$   $\ell = 0$

$$\mathbb{E}[T_\ell] = \sum_{k=0}^{\infty} k \cdot \Pr[T_\ell = k] = \sum_{k=0}^{\infty} \Pr[T_\ell > k]$$

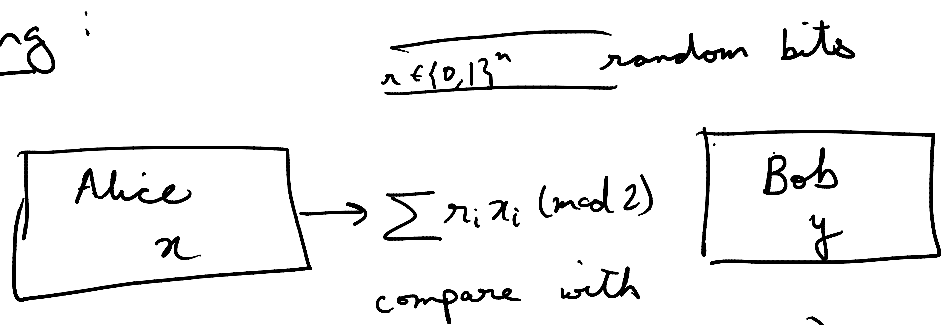
$$\leq \sum_{t=0}^{\infty} \left(\frac{1}{5}\right)^k = \frac{5}{4}$$

$$\Rightarrow \mathbb{E}[T] = O(n)$$

## Monte Carlo algorithms

- Recall: running time bounded for all inputs but chance of failure
- Simple MC algorithm above: find approximate median by choosing a random element!
- Another simple but very important application: checking polynomial identities
  - Check whether $p(x) \cdot q(x) = r(x)$ where $\deg(p)$, $\deg(q) \leq n$ and $\deg(r) \leq 2n$.
  - Instead of actually multiplying, evaluate $p, q$ and $r$ on random $x$ in $\{0, 1, 2, \ldots, 200n\}$.
  - $\Pr[(pq - r)(x) = 0] \leq \frac{2n}{200n} = 0.01$.

- Equality testing:



$\sum x_i y_i \pmod 2$.

If $x \neq y$, then unequal with probability $1/2$.

- Karger's algorithm
  - Min-Cut: min # of edges across a cut.
  - Algorithm:
    - While more than 2 vertices:
      - Pick a random edge and collapse
    - Output partition given by two clusters.
  - Fix min-cut $M$.
  - Min-cut size $\leq$ min-deg $\leq \dfrac{2|E|}{n}$
  - $\Pr[\text{random edge is in min-cut}] \leq \frac{2}{n}$.
  - $\Pr[\text{no edge selected is in min cut}]$

$$\geq \left(1-\frac{2}{n}\right)\cdot\left(1-\frac{2}{n-1}\right) \cdots\cdots \left(1-\frac{2}{3}\right)$$

$$= \frac{2}{n(n-1)}$$

  - $\Pr[M \text{ is output}] \geq \dfrac{1}{\binom{n}{2}}$ [Corollary: only $\leq \binom{n}{2}$ min cuts]