Sparse Fourier Transforms

Eric Price

UT Austin

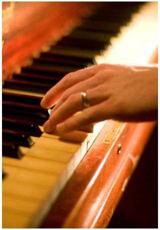
The Fourier Transform

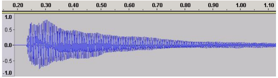
Conversion between time and frequency domains

Time Domain

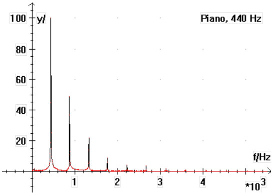
Frequency Domain

Fourier Transform ⟷





Displacement of Air

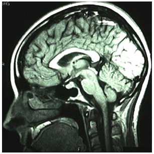


Concert A

The Fourier Transform is Ubiquitous

Audio

Video



Medical Imaging

Radar

GPS

Oil Exploration

Computing the Discrete Fourier Transform

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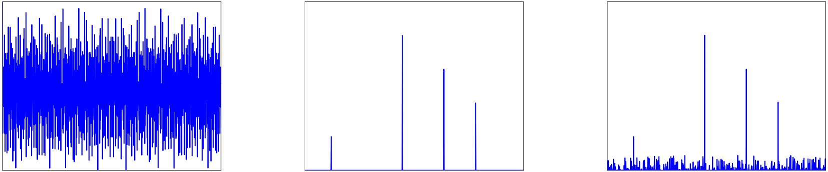
When can we compute the Fourier Transform in *sublinear* time?

Idea: Leverage *Sparsity*

Often the Fourier transform is dominated by a small number of peaks:



Time Signal

Frequency
(Exactly sparse)

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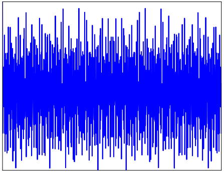

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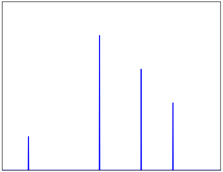

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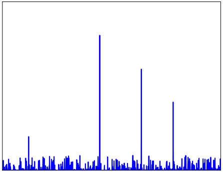

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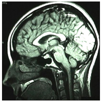
Sparsity is common:

Audio

Video



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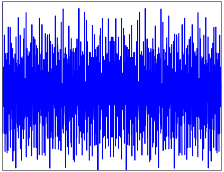

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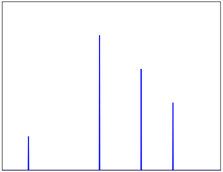

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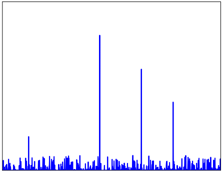

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Sparsity is common:

Goal of this talk: *sparse* Fourier transforms

Faster Fourier Transform on sparse data.

Recent Theory and Applied Work

- Sparse Fourier Transform in the Discrete Setting
  - Gilbert-Guha-Indyk-Muthukrishnan-Strauss, 02
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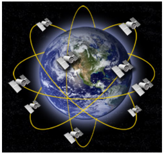
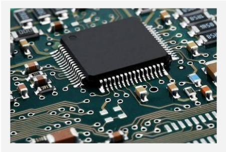
- Applications



Faster GPS ... Fourier ...
Hassanieh et al.
MOBICOM'12



... Fourier ... Chip ...
Abari et al.
ISSCC'12

... Chemical ... Imaging ...
Andronesi et al.
ENC'14

Light ... Continuous Fourier...
Shi et al.
SIGGRAPH'15

Kinds of discrete Fourier transform

- 1d Fourier transform: $x \in \mathbb{C}^n$, $\omega = e^{2\pi \mathbf{i}/n}$, want

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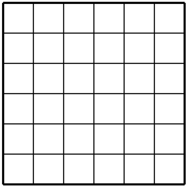

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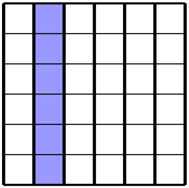

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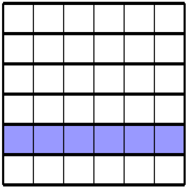

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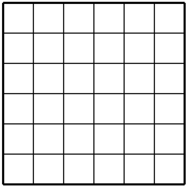

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- Hadamard transform: $x \in \mathbb{C}^{2 \times 2 \times \cdots \times 2}$:

$$\widehat{x}_i = \sum_{j}^{n} (-1)^{\langle i,j \rangle} x_j$$

Generic Algorithm Outline

- Goal: given access to x, compute $\overline{x} \approx \widehat{x}$
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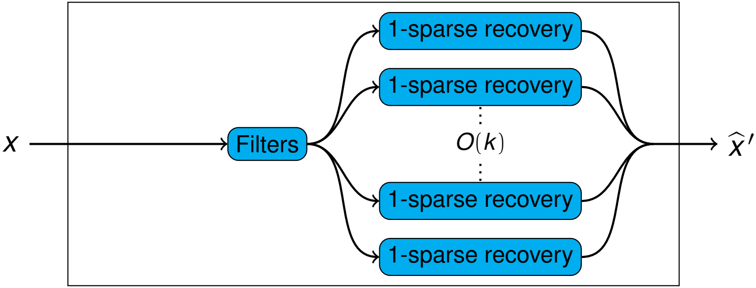

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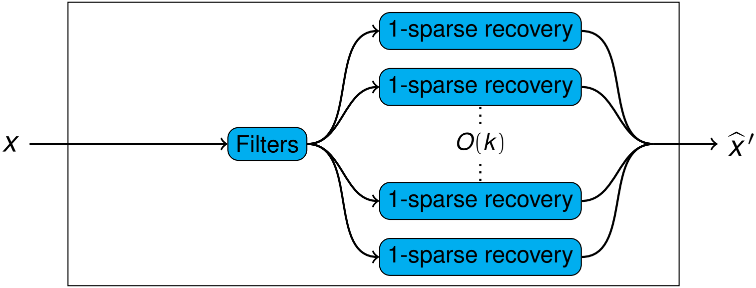

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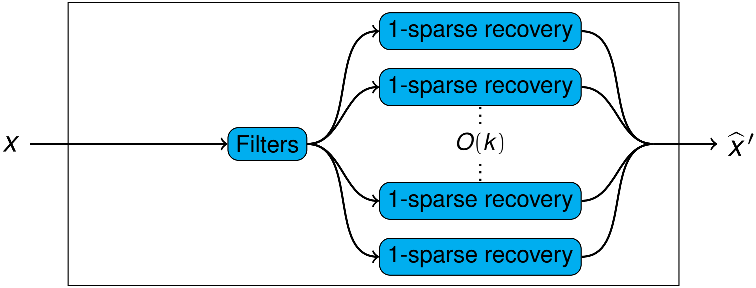

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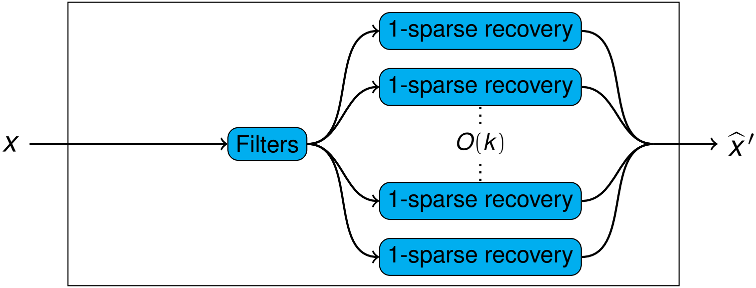

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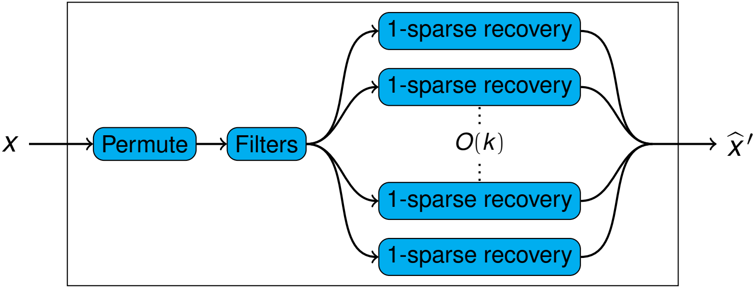

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3. Finds "most" of signal; repeat on residual

Talk Outline

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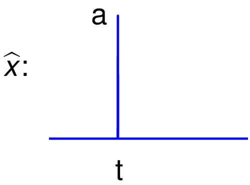
Algorithm for $k = 1$: one dimension, exact case

Lemma

We can compute a 1-sparse $\widehat{x}$ in $O(1)$ time.



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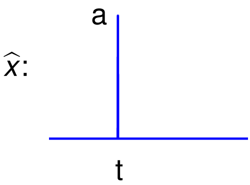

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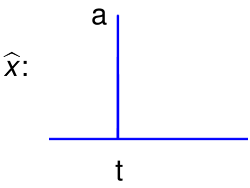

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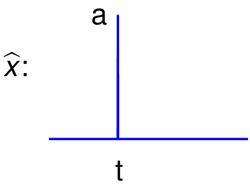

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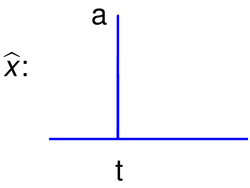

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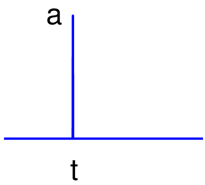

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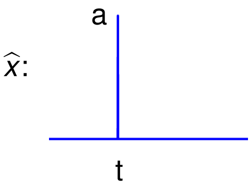

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- (Related to OFDM, Prony's method, matrix pencil.)

Algorithm for $k = 1$: one dimension, approximate case

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Suppose $\widehat{x}$ is approximately 1-sparse:

$$|\widehat{x}_t| / \|\widehat{x}\|_2 \geqslant 90\%.$$

Then we can recover it with $O(\log n)$ samples and $O(\log^2 n)$ time.

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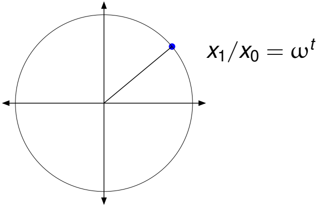

- With exact sparsity: $\log n$ bits in a single measurement.

Algorithm for $k = 1$: one dimension, approximate case

Lemma

Suppose $\widehat{x}$ is approximately 1-sparse:

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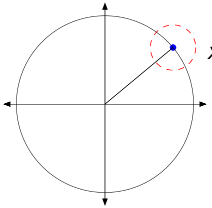

- With exact sparsity: $\log n$ bits in a single measurement.
- With noise: only constant number of useful bits.

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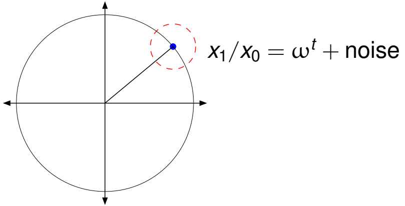

- With exact sparsity: $\log n$ bits in a single measurement.
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- Choose $\Theta(\log n)$ time shifts c to recover i.

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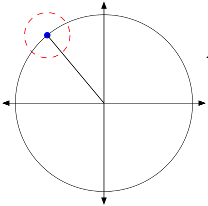

$$x_{c_2}/x_0 = \omega^{c_2 t} + \text{noise}$$

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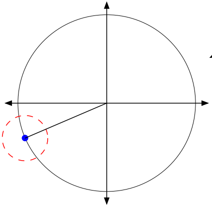

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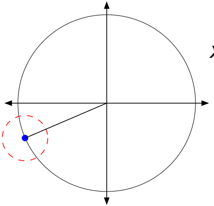

- With exact sparsity: $\log n$ bits in a single measurement.
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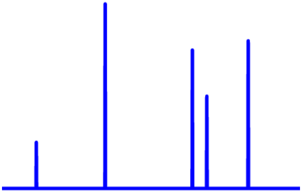
- Choose $\Theta(\log n)$ time shifts c to recover i.
- Error correcting code with efficient recovery $\implies$ lemma. ∎

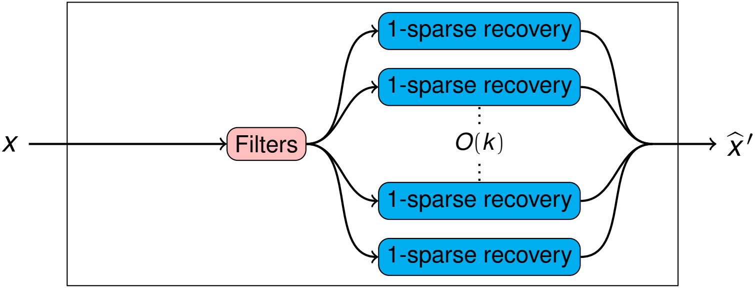
Talk Outline

1. Algorithm for $k = 1$
2. Reducing k to 1
3. Putting it together
4. Continuous setting

Algorithm for general k

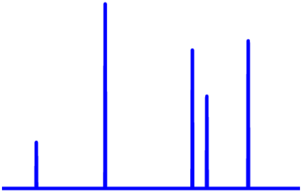
- Reduce general k to $k = 1$.

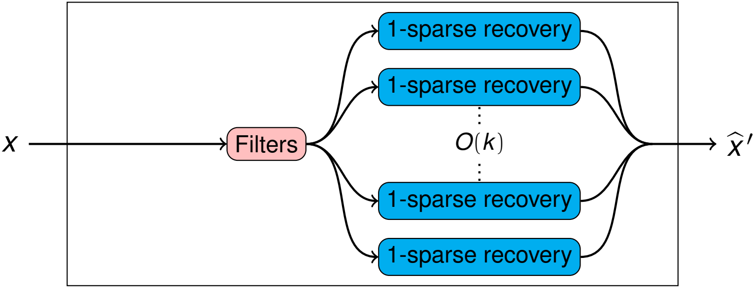




Algorithm for general k

- Reduce general k to $k = 1$.
- "Filters": partition frequencies into $O(k)$ buckets.





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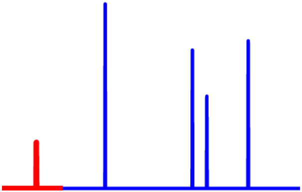

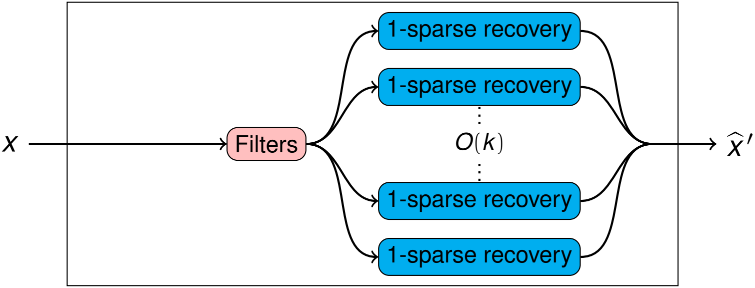

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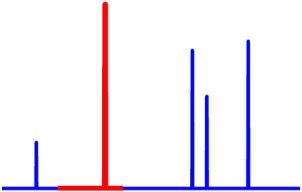

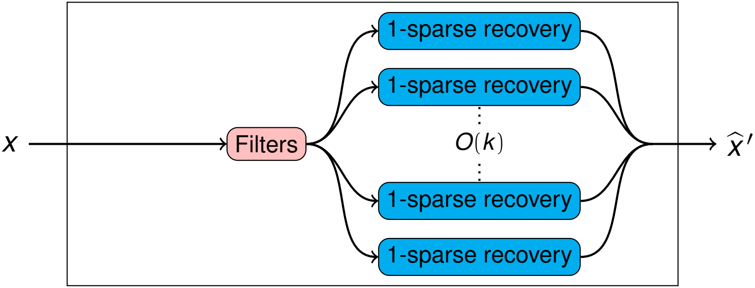

Algorithm for general k

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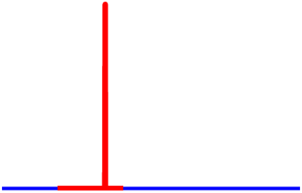

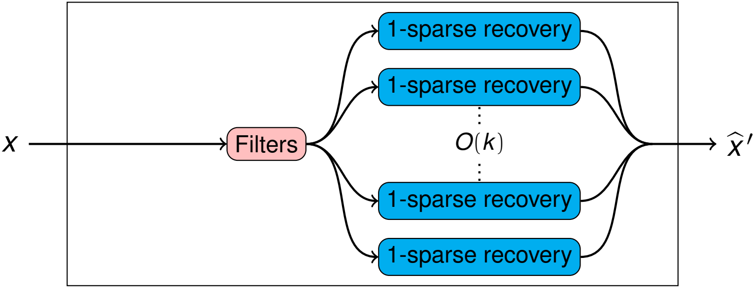

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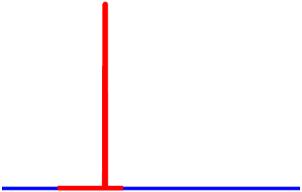

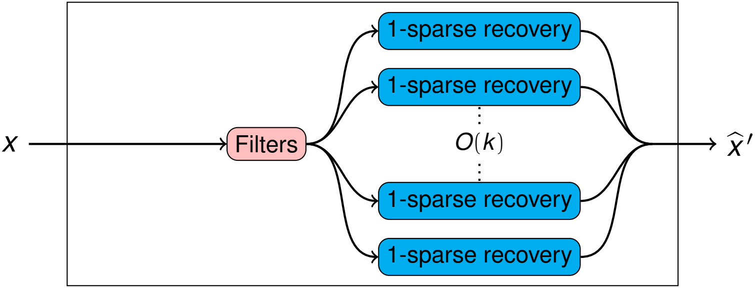

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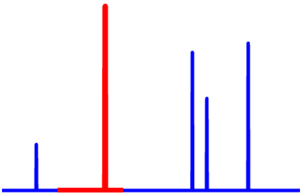

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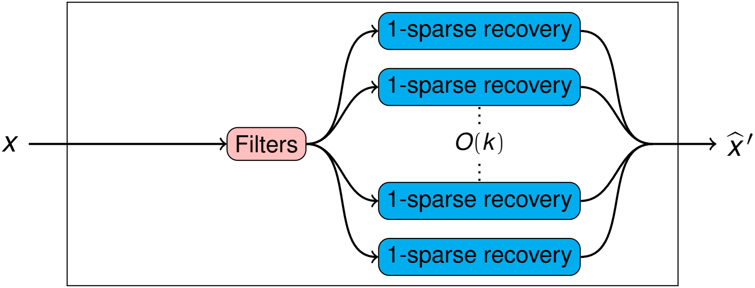

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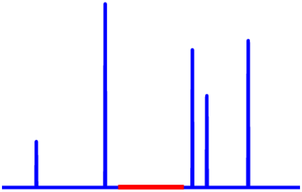

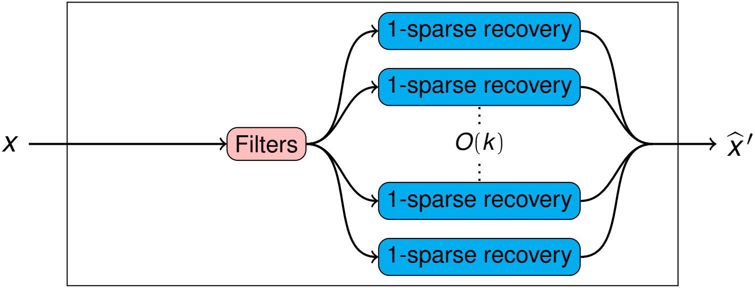

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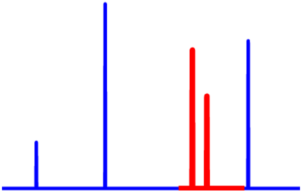

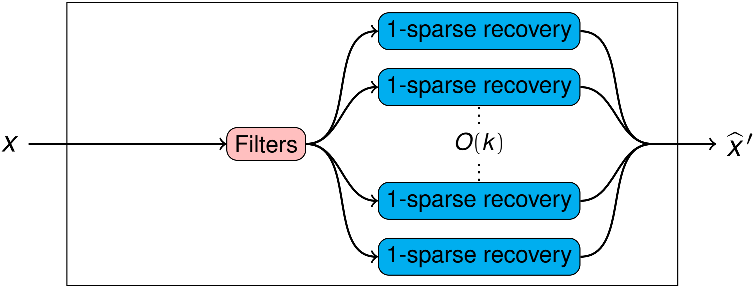

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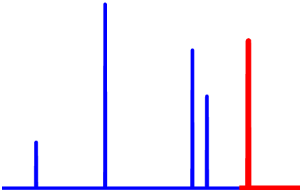

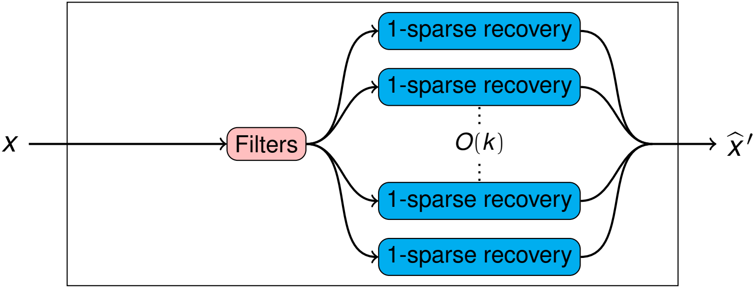

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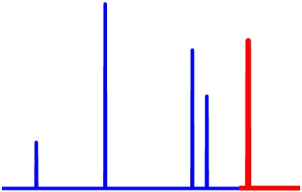

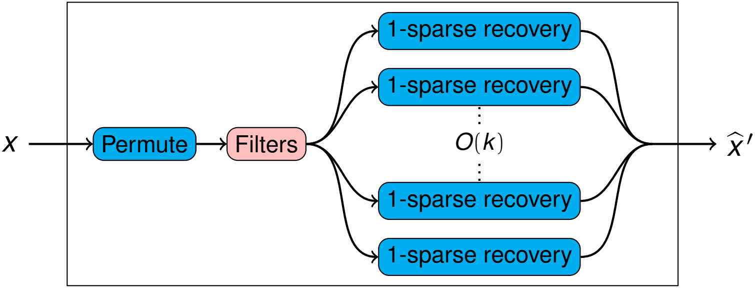

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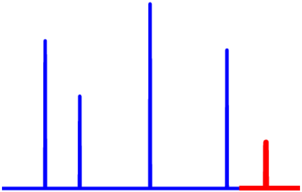

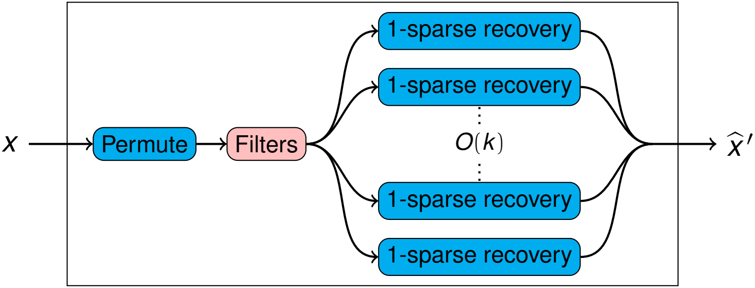

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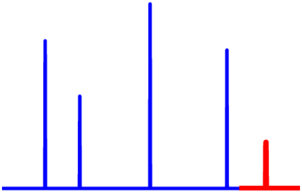

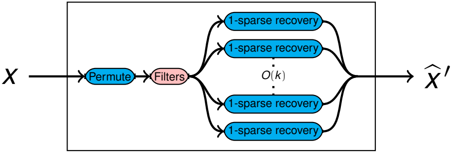

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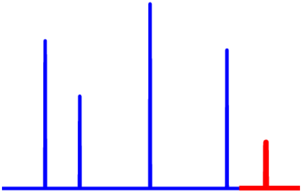
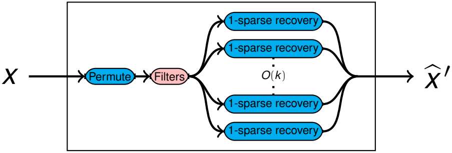


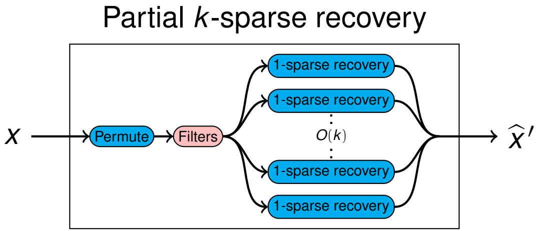


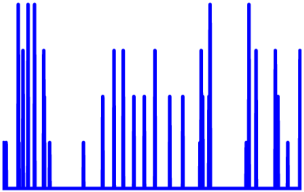
Recovers *most* of $\widehat{x}$:

Lemma (Partial sparse recovery)

In $O(k \log n)$ expected time, we can compute an estimate $\widehat{x}'$ such that $\widehat{x} - \widehat{x}'$ is $k/2$-sparse.

Going from finding most coordinates to finding all $\widehat{x}$





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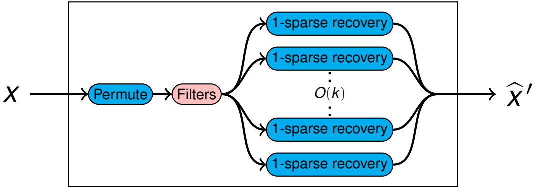
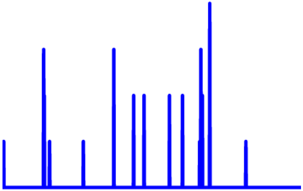


$\widehat{x} - \widehat{x}'$



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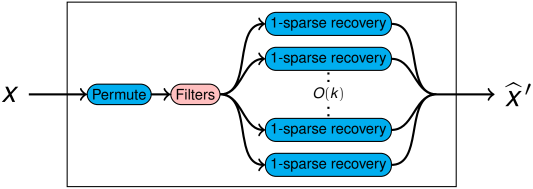

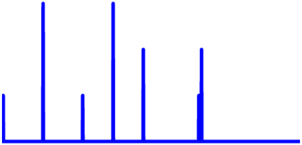

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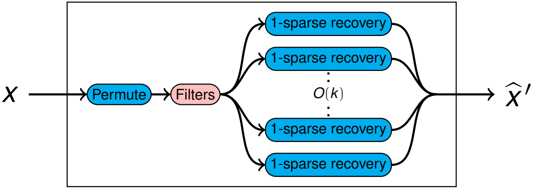
Repeat, $k \to k/2 \to k/4 \to \cdots$

Going from finding most coordinates to finding all

Partial k-sparse recovery



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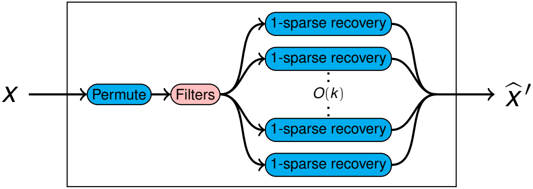

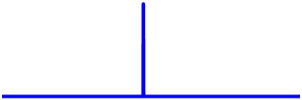

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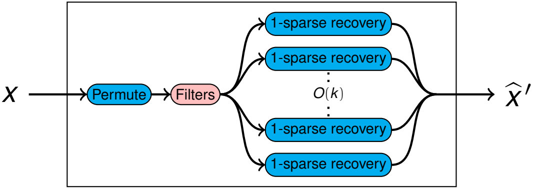

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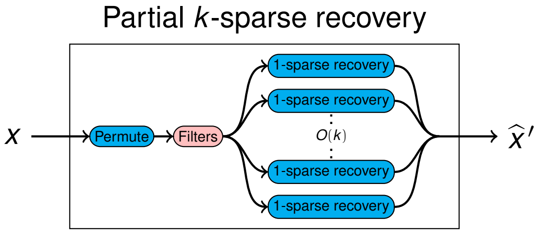

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Theorem

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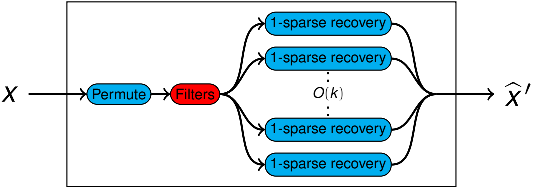
Going from finding most coordinates to finding all $\widehat{x} - \widehat{x}'$

Partial k-sparse recovery



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Talk Outline

1. Algorithm for $k = 1$
2. Reducing k to 1
3. Putting it together
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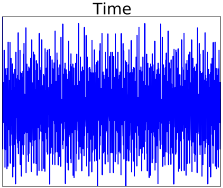
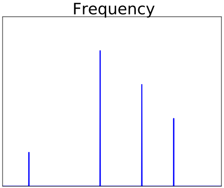
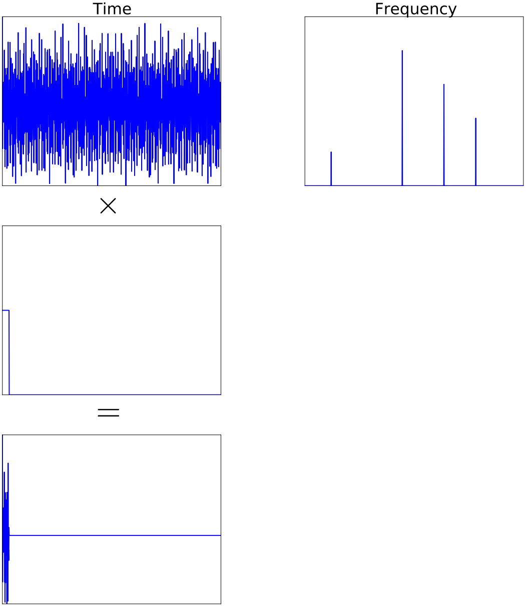
How can you hope for sublinear time?





n-dimensional DFT:
$O(n \log n)$
$x \to \widehat{x}$

How can you hope for sublinear time?



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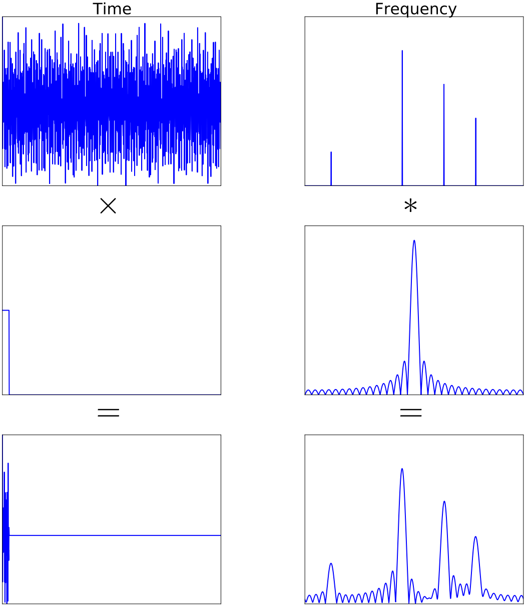

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n-dimensional DFT of first
k terms: $O(n \log n)$
$x \cdot \text{rect} \to \widehat{x} * \text{sinc}$.

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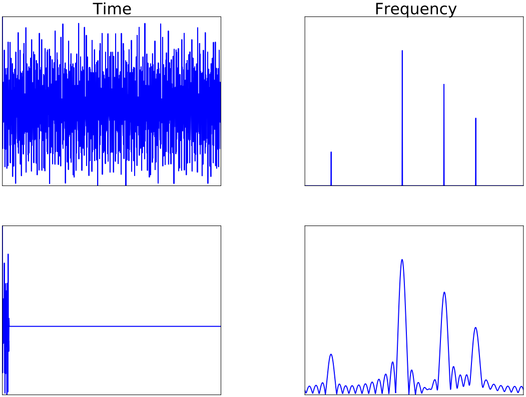

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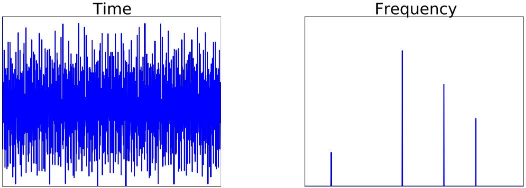
$x \cdot \text{rect} \rightarrow \widehat{x} * \text{sinc}$.

How can you hope for sublinear time?



n-dimensional DFT:
$O(n \log n)$
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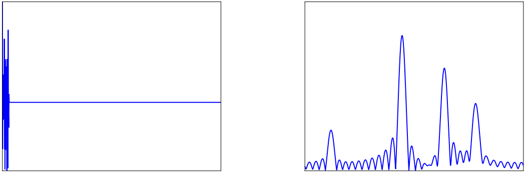

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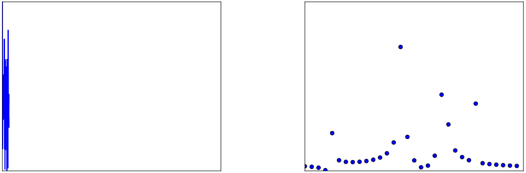

k-dimensional DFT of first k terms: $O(B \log B)$
$\text{alias}(x \cdot \text{rect}) \to \text{subsample}(\widehat{x} * \text{sinc})$.

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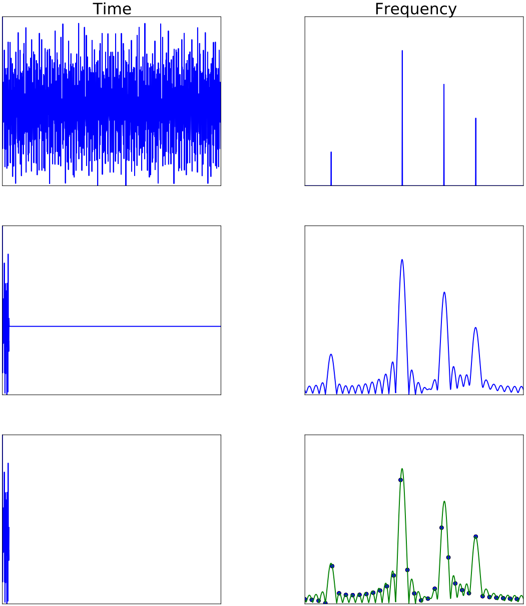

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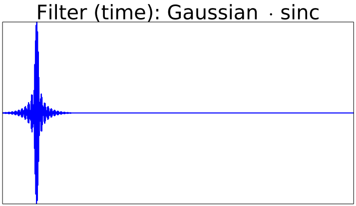
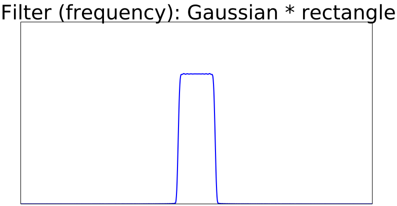
Use a better filter

GMS05, HIKP12, IKP14, IK14





- Filter: sparse F for which $\widehat{F}$ is large on an *interval*.

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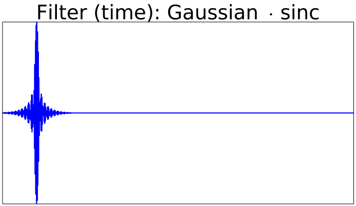
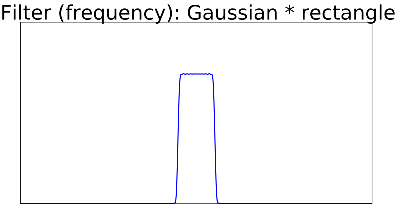
GMS05, HIKP12, IKP14, IK14





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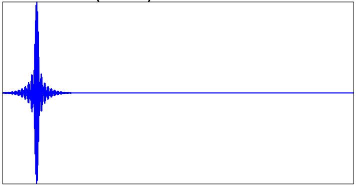
$$x'_i = x_{\sigma i} \implies \widehat{x}_i = \widehat{x}_{\sigma^{-1} i}$$

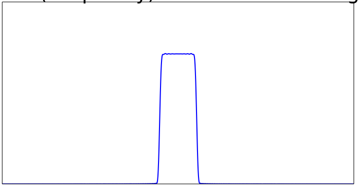
Use a better filter

GMS05, HIKP12, IKP14, IK14

Filter (time): Gaussian $\cdot$ sinc



Filter (frequency): Gaussian * rectangle



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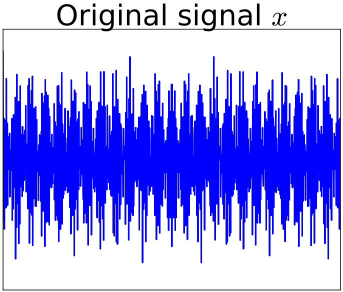
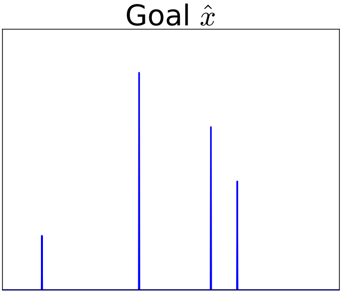
- Allows us to convert worst case to random case.

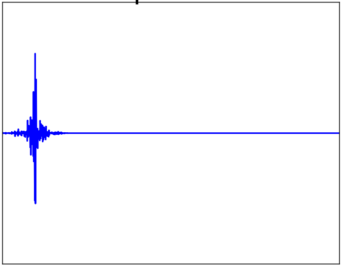
Algorithm for *exactly sparse* signals



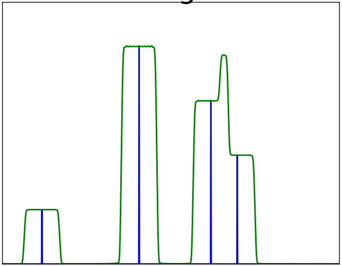


Algorithm for *exactly sparse* signals

Computed $F \cdot x$



Filtered signal $\widehat{F} * \widehat{x}$



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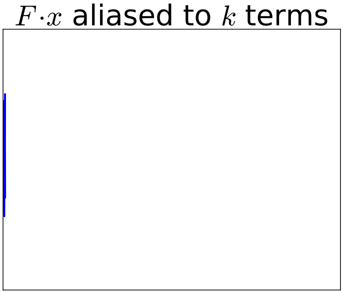
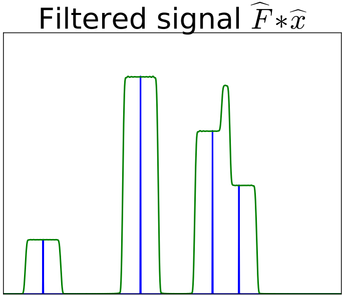
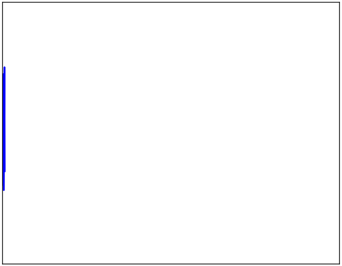


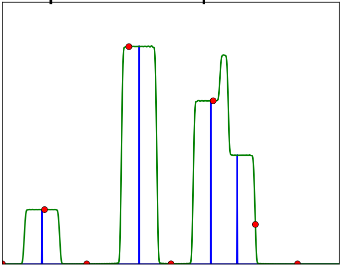


Algorithm for *exactly sparse* signals

$F \cdot x$ aliased to k terms



Computed samples of $\widehat{F} * \widehat{x}$



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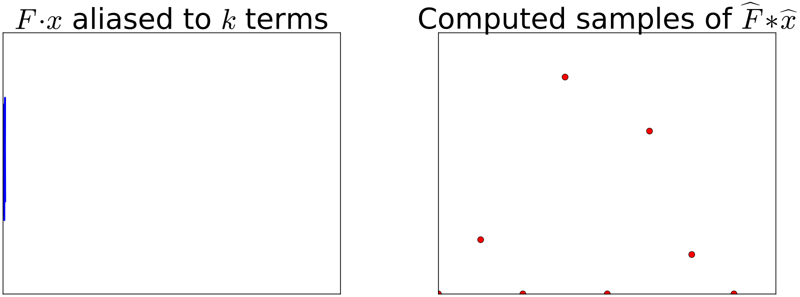

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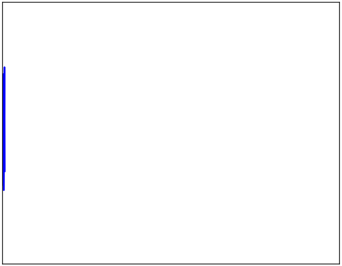
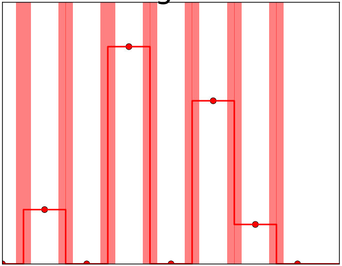


Knowledge about $\hat{x}$



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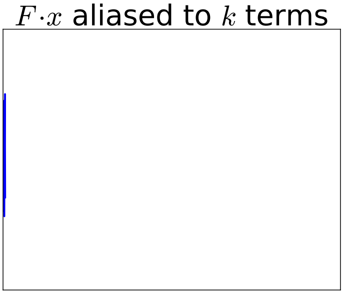

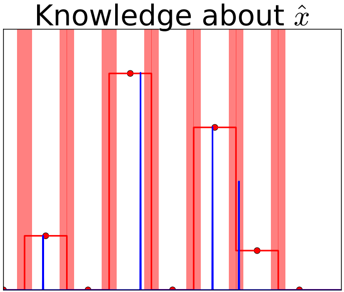

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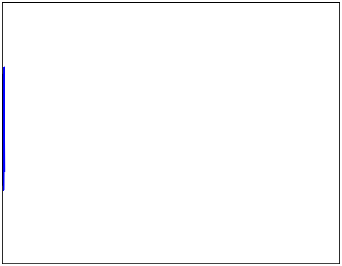

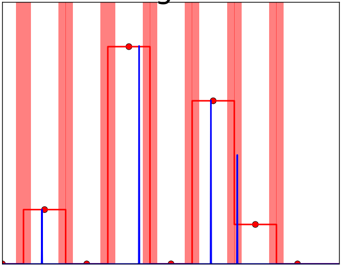

Lemma

If t is isolated in its bucket and in the "super-pass" region, the value b we compute for its bucket satisfies

$$b = \widehat{x}_t.$$

Computing the b for all $O(k)$ buckets takes $O(k \log n)$ time.

Algorithm

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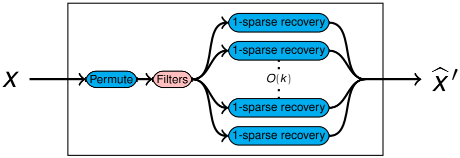

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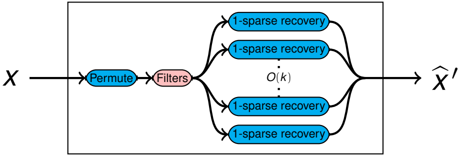

- Repeat $k \to k/2 \to k/4 \to \cdots$

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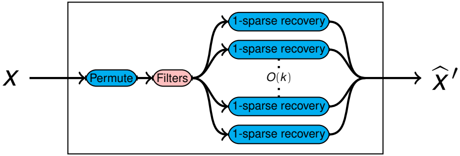

- Repeat $k \to k/2 \to k/4 \to \cdots$
- $O(k \log n)$ time sparse Fourier transform. ■

Summary (DFT setting)

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  - Optimal samples [IK '14] OR optimal time [HIKP '12] OR $\log^c \log n$-competitive mixture [IKP '14].

Talk Outline

1. Algorithm for $k = 1$
2. Reducing k to 1
3. Putting it together
4. Continuous setting

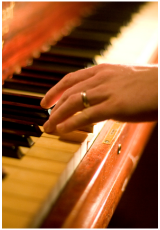
The *Continuous* Fourier Transform

Conversion between time and frequency domains

Time Domain

Fourier Transform ⟷

Frequency Domain



- The Fourier Transform $\widehat{x}$ of an integrable function $x : \mathbb{R} \to \mathbb{C}$ is

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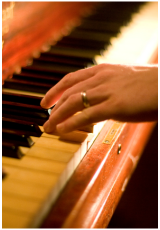
Fourier Transform $\longleftrightarrow$



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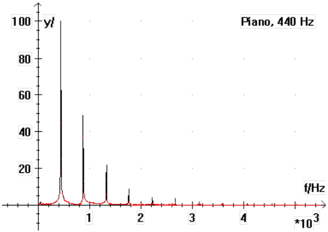
- The Inverse Transform is:

$$x(t) = \int_{-\infty}^{+\infty} \widehat{x}(f) e^{2\pi \mathbf{i} f t} \mathrm{d}f$$

Why Continuous?



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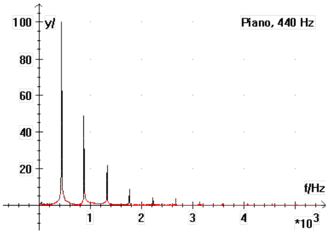

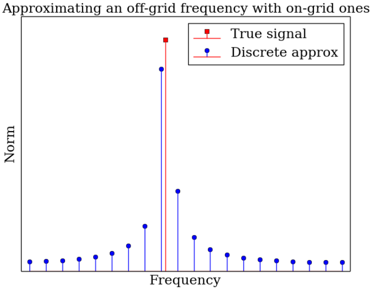

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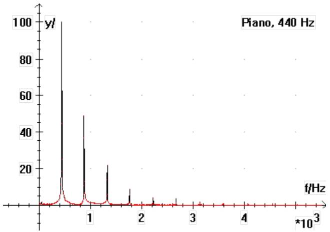

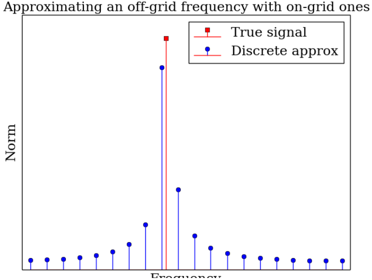

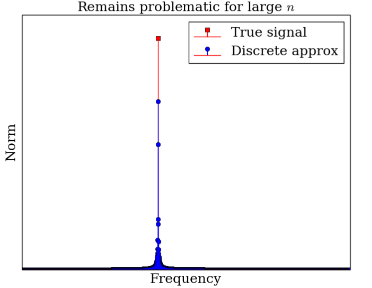

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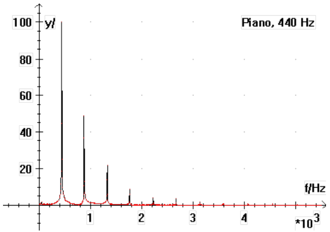
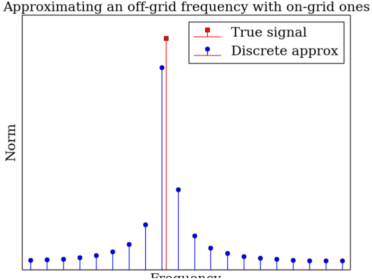
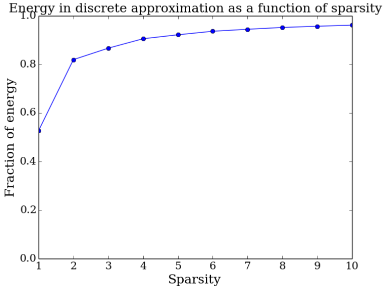
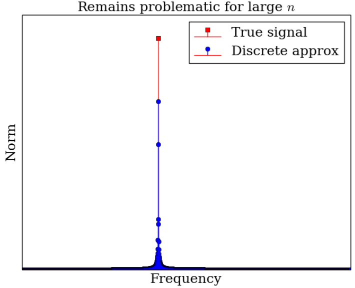


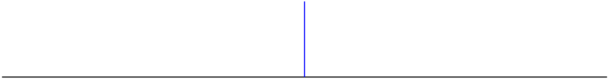




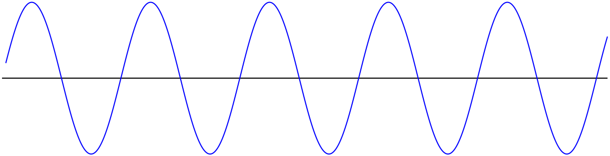


Thought Experiments

Frequency



Time



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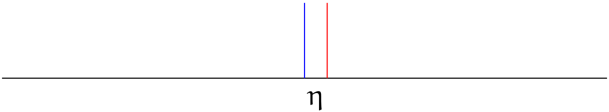

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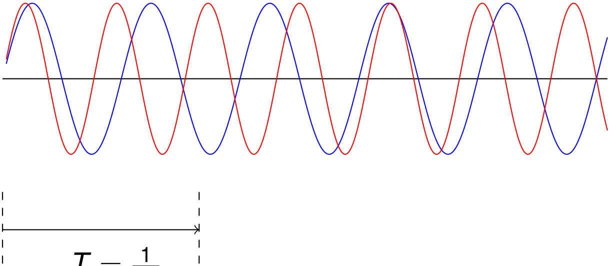
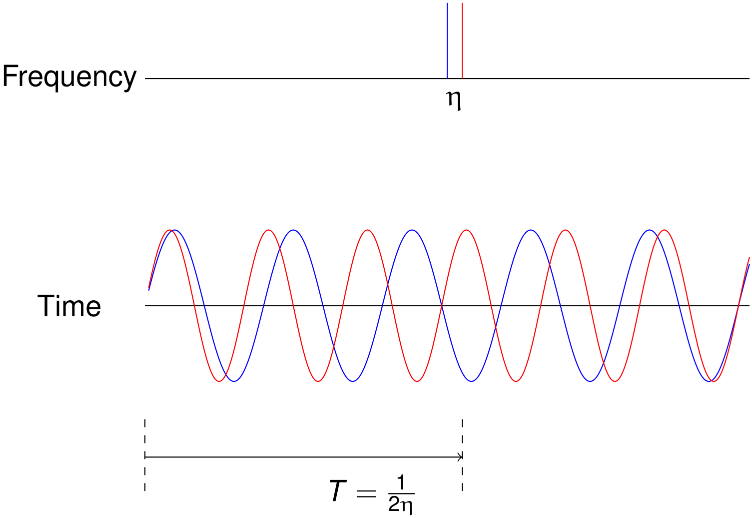
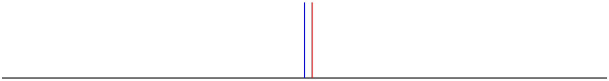


$T = \frac{1}{2\eta}$

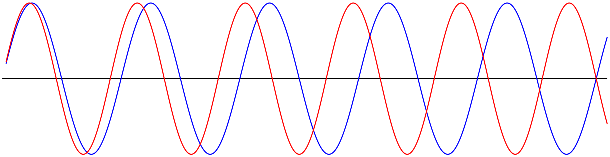
Thought Experiments



Thought Experiments

Frequency 

η

Time 

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Thought Experiments

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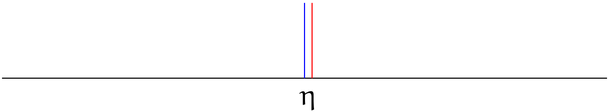

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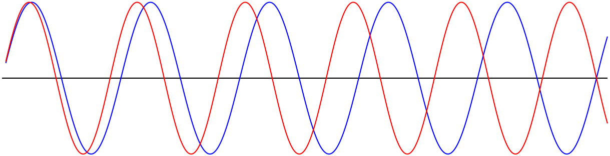


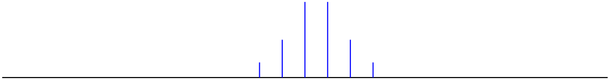
$T = \frac{1}{2\eta}$

inifinitely small η

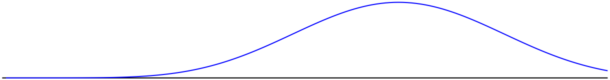
$T > 1000$ years

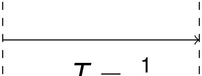
Thought Experiments 2

$$x(t) = \sin^{k-1}(\eta t)$$

Frequency  

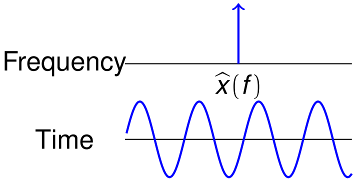
η

Time  



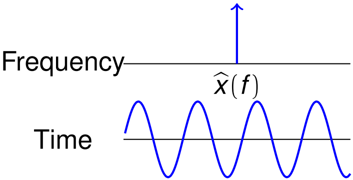
$T = \frac{1}{2\eta}$

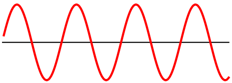
What Guarantee Do We Want?



$x(t) = e^{-2\pi \mathbf{i} f t}$

What Guarantee Do We Want?

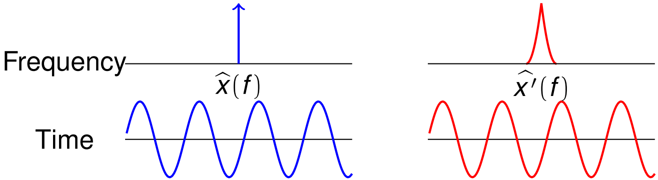




$x(t) = e^{-2\pi \mathbf{i} f t}$

$x'(t) = x(t) \cdot e^{-t/10000}$

What Guarantee Do We Want?



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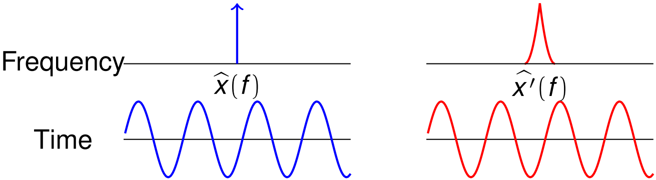

$x(t) = e^{-2\pi \mathbf{i} f t}$

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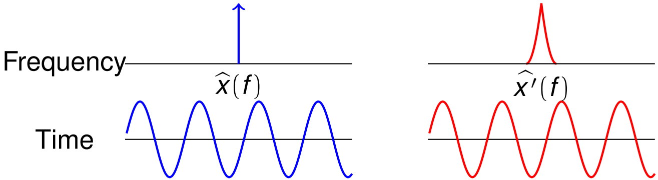
Discrete FT $\|\widehat{x'} - \widehat{x}\|_2 \lesssim \min_{k\text{-sparse}\widehat{x}_k} \|\widehat{x} - \widehat{x}_k\|_2$

What Guarantee Do We Want?



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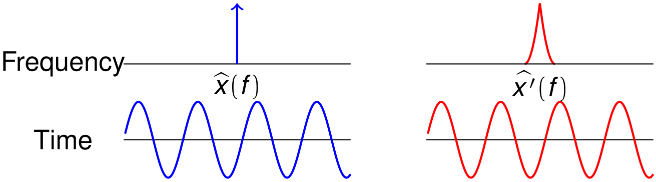

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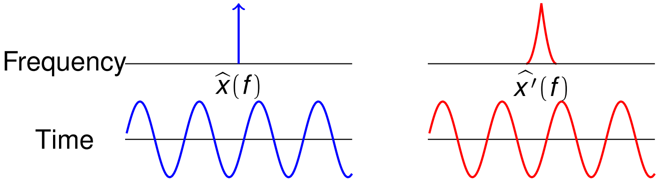
DFT preserve the ℓ_2 norm

What Guarantee Do We Want?



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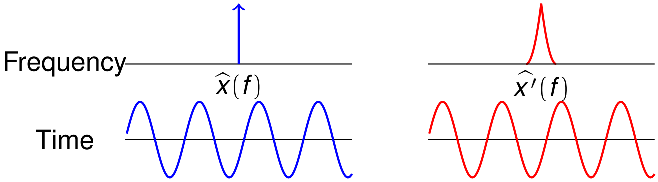
What Guarantee Do We Want?



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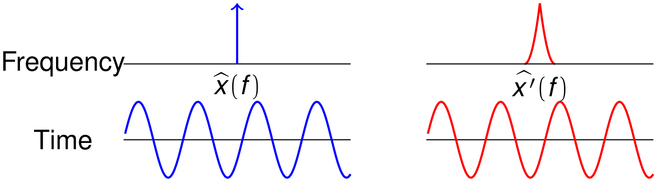
$$\frac{1}{T}\int_0^T |x'(t) - x(t)|^2 \mathrm{d}t \lesssim \min_{k\text{-sparse } \widehat{x}_k(t)} \frac{1}{T}\int_0^T |x(t) - x_k(t)|^2 \mathrm{d}t$$

What Guarantee Do We Want?



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$$\boxed{\frac{1}{T}\int_0^T |x'(t) - x(t)|^2 \mathrm{d}t \lesssim \min_{k\text{-sparse } \widehat{x}_k(t)} \frac{1}{T}\int_0^T |x(t) - x_k(t)|^2 \mathrm{d}t}$$

Guarantee

- Sample from $x(t)$, which is approximated by a k-Fourier sparse $x_k(t)$ with η frequency separation.
- We recover an $x'(t)$ such that

$$\mathop{\mathbb{E}}_{t \in [0,T]} |x'(t) - x(t)|^2 \lesssim \mathop{\mathbb{E}}_{t \in [0,T]} |x(t) - x_k(t)|^2$$

- As long as:
  - $T \geqslant O(\frac{\log^2(FT)}{\eta})$
  - Time, # samples $\geqslant O(k \log(FT) \log^2(k))$.

Previous Works and Our Results

Algorithm	Duration	Robust	Sample/Time
BCGLS, 12	$k\cdot$optimal	poor	sublinear
Moitra, 15	optimal	poly(k)	linear
Ours	$\log^2(k)\cdot$optimal	$O(1)$	sublinear

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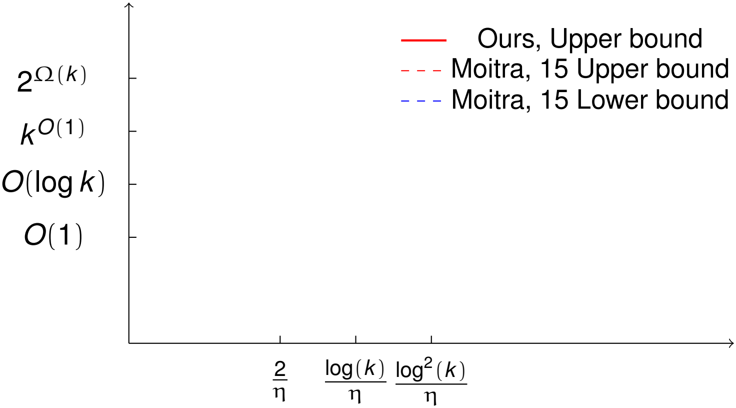

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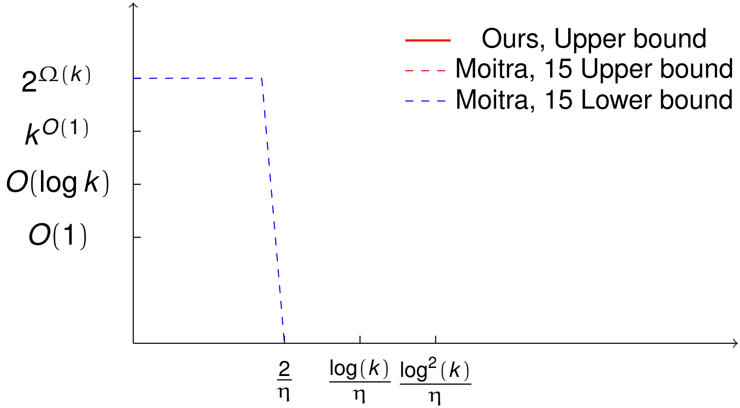

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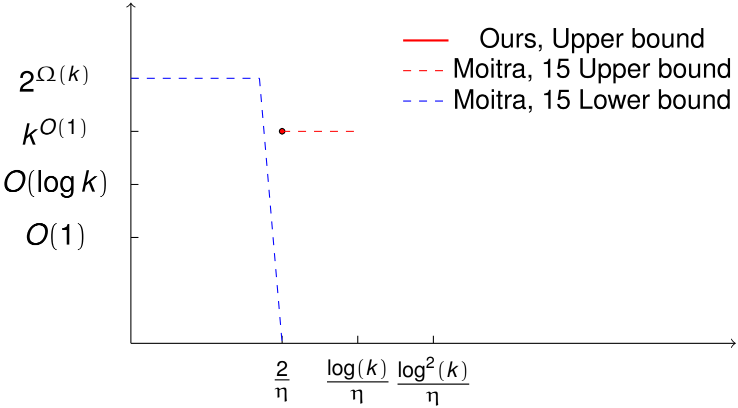

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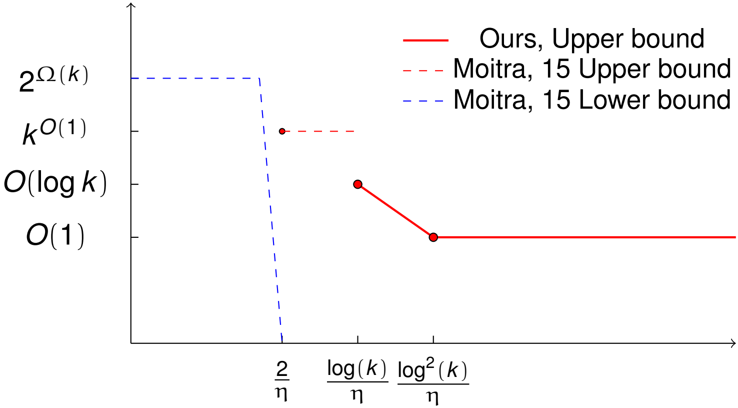
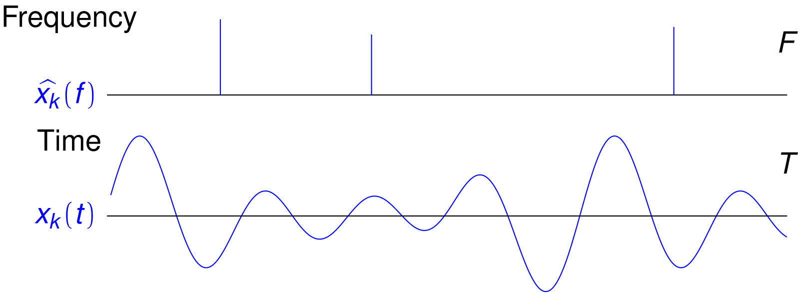
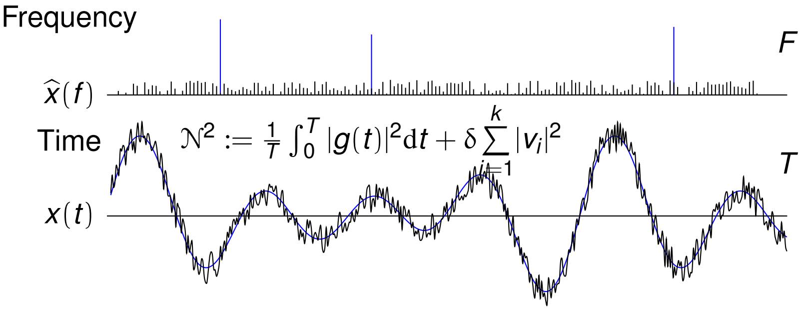


Main Results

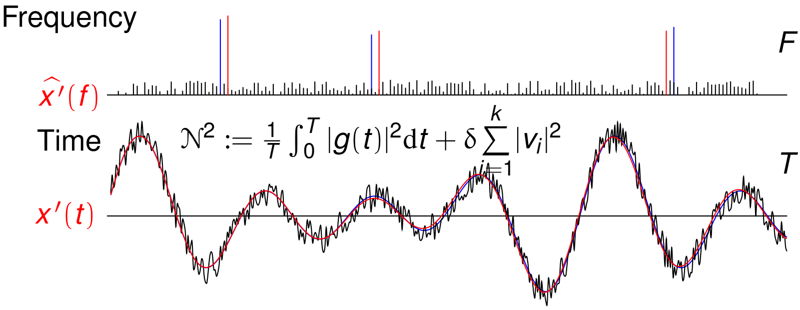


Main Results



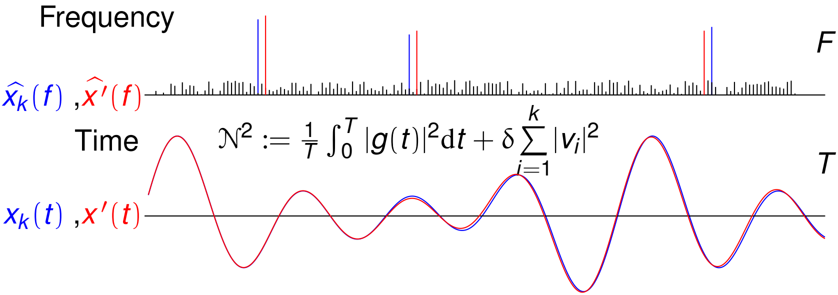
$$\mathcal{N}^2 := \frac{1}{T}\int_0^T |g(t)|^2 \mathrm{d}t + \delta \sum_{i=1}^{k} |v_i|^2$$

Main Results



$$\mathcal{N}^2 := \frac{1}{T}\int_0^T |g(t)|^2 \mathrm{d}t + \delta \sum_{i=1}^{k} |v_i|^2$$

Main Results



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Tone Estimation

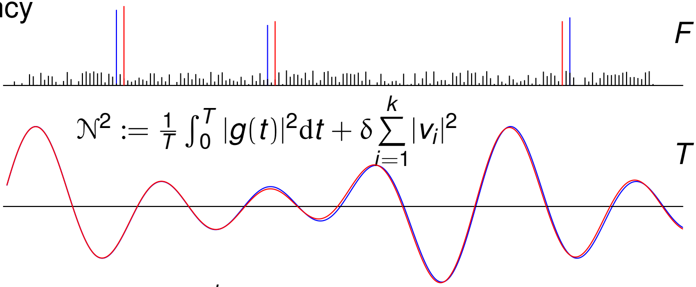
Main Results

Frequency

$\widehat{x_k}(f)$, $\widehat{x'}(f)$

Time

$x_k(t)$, $x'(t)$



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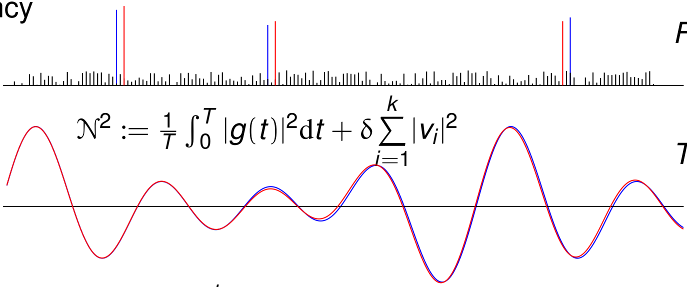
Tone Estimation $$\frac{1}{T}\sum_{i=1}^{k}\int_0^T |\, v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}\,|^2 \mathrm{d}t \lesssim \mathcal{N}^2$$

Main Results

Frequency

$\widehat{x_k}(f)$, $\widehat{x'}(f)$



Time

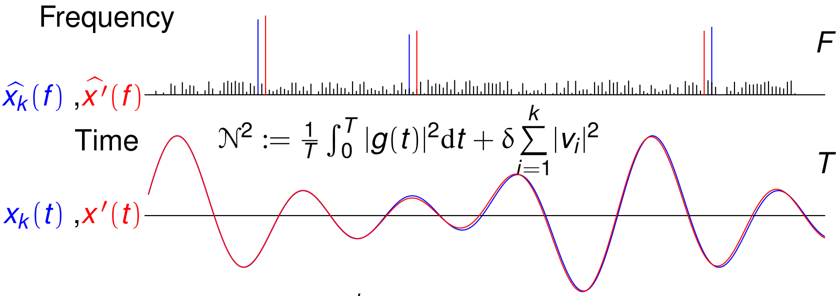
$$\mathcal{N}^2 := \frac{1}{T}\int_0^T |g(t)|^2 \mathrm{d}t + \delta \sum_{i=1}^{k} |v_i|^2$$

$x_k(t)$, $x'(t)$

Tone Estimation $$\frac{1}{T}\sum_{i=1}^{k}\int_0^T \left| v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t} \right|^2 \mathrm{d}t \lesssim \mathcal{N}^2$$

Frequency Estimation

Main Results



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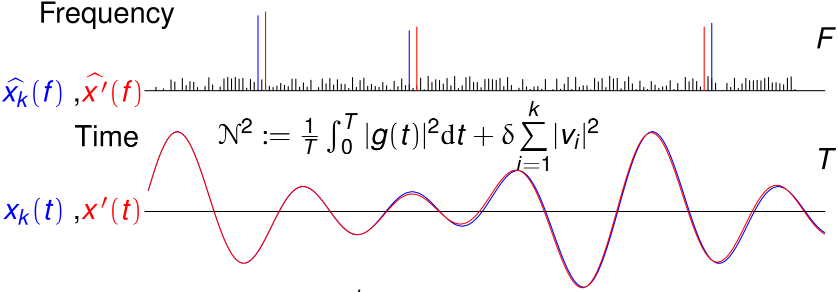
Frequency Estimation $|f_i - f_i'| \lesssim \frac{1}{T\rho}$, $\rho^2 := SNR := \frac{\sum_i |v_i|^2}{\mathcal{N}^2}$

Main Results



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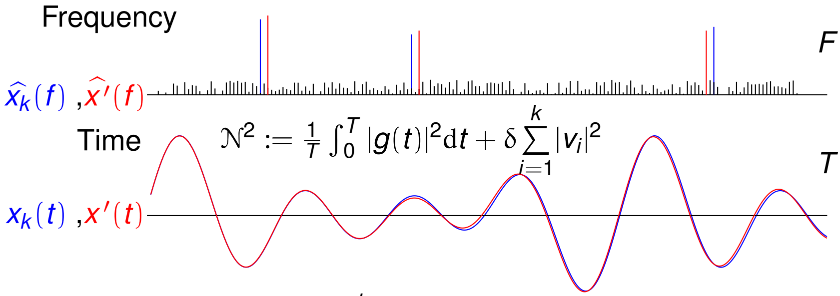
Signal Estimation

Main Results



$$\mathcal{N}^2 := \frac{1}{T}\int_0^T |g(t)|^2 \mathrm{d}t + \delta \sum_{i=1}^{k} |v_i|^2$$

Tone Estimation $\frac{1}{T}\sum_{i=1}^{k}\int_0^T |\, v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t} \,|^2 \mathrm{d}t \lesssim \mathcal{N}^2$

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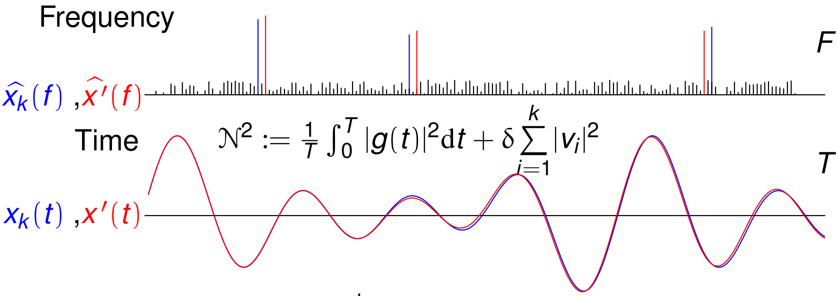
Main Results



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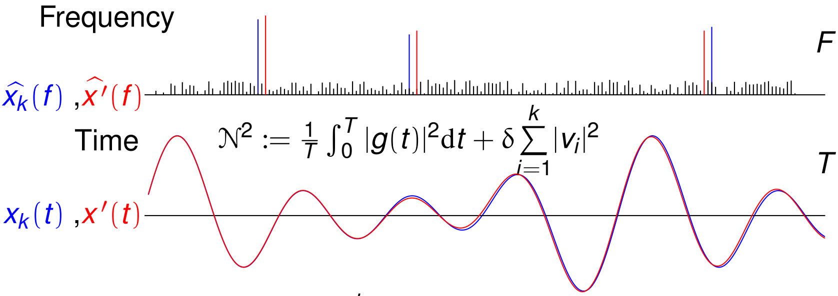
Duration

Main Results



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Duration $T = \frac{\log(k)}{\eta}$, $T = \frac{\log^2(k)}{\eta}$

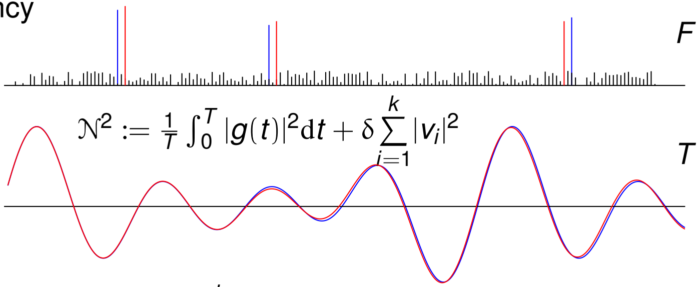
Main Results

Frequency

$\widehat{x_k}(f)$, $\widehat{x'}(f)$

Time

$x_k(t)$, $x'(t)$



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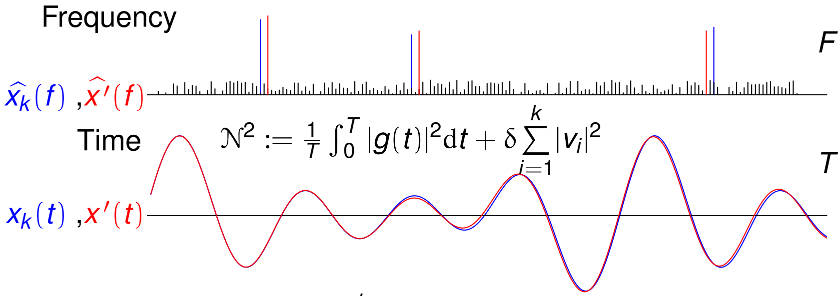
Duration $\quad T = \frac{\log(k)}{\eta}$, $T = \frac{\log^2(k)}{\eta}$

Samples/Time

Main Results



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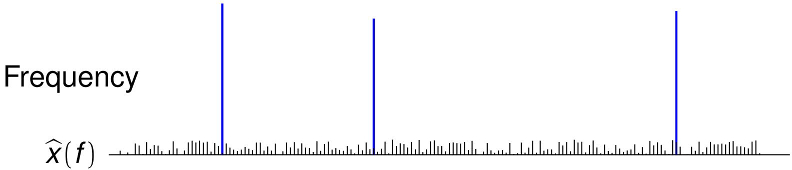
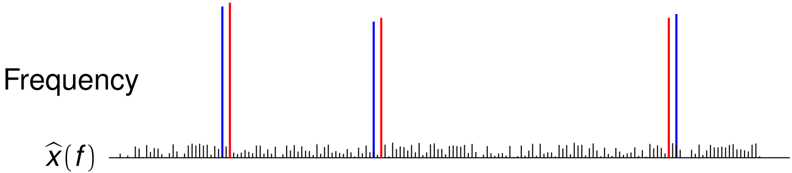
Duration $\quad T = \frac{\log(k)}{\eta}, T = \frac{\log^2(k)}{\eta}$

Samples/Time $\quad O(k \log(FT) \log^2(k))$

Tone Estimation to Signal Estimation

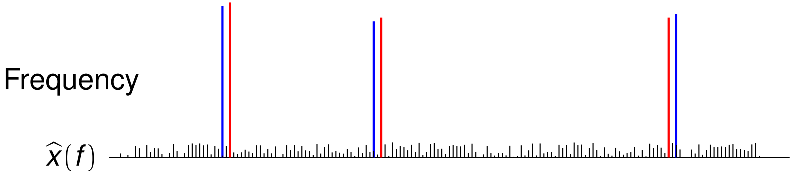


Tone Estimation to Signal Estimation



Tone Estimation to Signal Estimation

Define $\Delta_i(t) = a_i(t) - a_i'(t) \;= v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$



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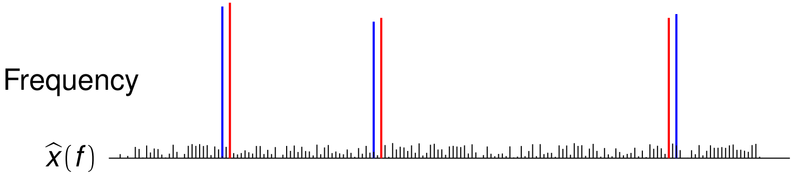
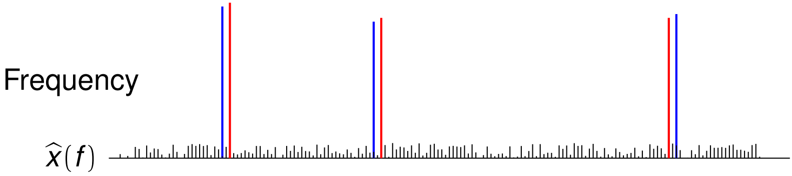


$\frac{1}{T}\int_0^T |a_1(t) - a_1'(t)|^2 \mathrm{d}t$

Tone Estimation to Signal Estimation

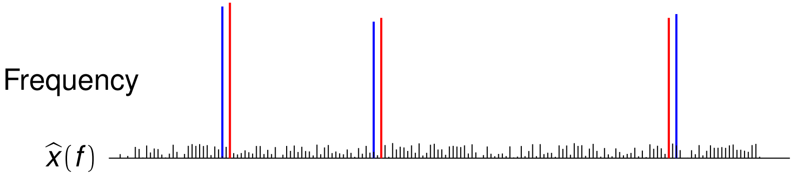
Define $\Delta_i(t) = a_i(t) - a'_i(t) = v_i e^{2\pi \mathbf{i} f_i t} - v'_i e^{2\pi \mathbf{i} f'_i t}$



$\frac{1}{T}\int_0^T |a_1(t) - a'_1(t)|^2 \mathrm{d}t \quad \frac{1}{T}\int_0^T |a_2(t) - a'_2(t)|^2 \mathrm{d}t$

Tone Estimation to Signal Estimation

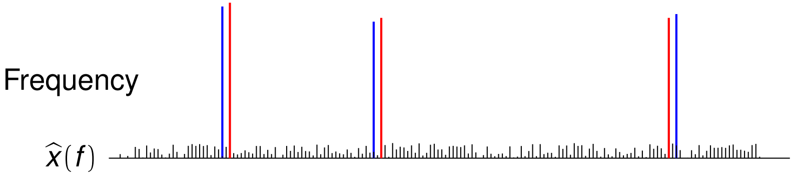
Define $\Delta_i(t) = a_i(t) - a_i'(t) \ = v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$



$\frac{1}{T}\int_0^T |a_1(t) - a_1'(t)|^2 \mathrm{d}t \quad \frac{1}{T}\int_0^T |a_2(t) - a_2'(t)|^2 \mathrm{d}t \quad \frac{1}{T}\int_0^T |a_3(t) - a_3'(t)|^2 \mathrm{d}t$

Tone Estimation to Signal Estimation

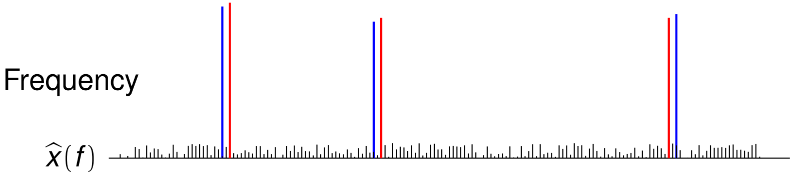
Define $\Delta_i(t) = a_i(t) - a_i'(t) = v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$



$\frac{1}{T}\int_0^T |a_1(t) - a_1'(t)|^2 \mathrm{d}t + \frac{1}{T}\int_0^T |a_2(t) - a_2'(t)|^2 \mathrm{d}t + \frac{1}{T}\int_0^T |a_3(t) - a_3'(t)|^2 \mathrm{d}t$

Tone Estimation to Signal Estimation

Define $\Delta_i(t) = a_i(t) - a_i'(t) \; = v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$



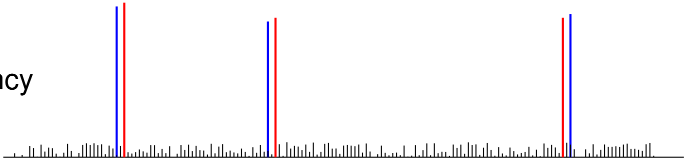
$\sum_{i=1}^{k} \frac{1}{T} \int_0^T |a_i(t)$ - $a_i'(t)|^2 \mathrm{d}t$

Tone Estimation to Signal Estimation

Define $\Delta_i(t) = a_i(t) - a_i'(t) = v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$

Frequency

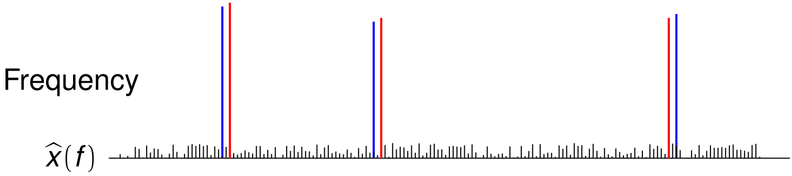
$\widehat{x}(f)$



$$\sum_{i=1}^{k} \frac{1}{T} \int_0^T |\Delta_i(t)|^2 \mathrm{d}t$$

Tone Estimation to Signal Estimation

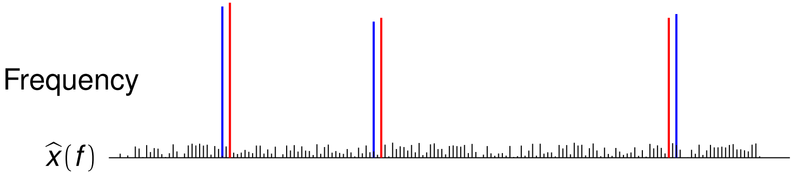
Define $\Delta_i(t) = a_i(t) - a_i'(t) \; = v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$



$\mathcal{N}^2 \gtrsim \sum_{i=1}^{k} \frac{1}{T} \int_0^T |\Delta_i(t)|^2 \mathrm{d}t$ Tone Estimation

Tone Estimation to Signal Estimation

Define $\Delta_i(t) = a_i(t) - a_i'(t) = v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$

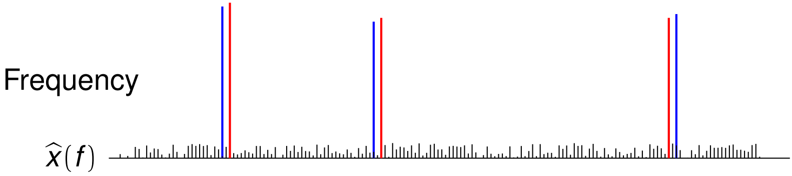


$$\mathcal{N}^2 \gtrsim \sum_{i=1}^{k} \frac{1}{T} \int_0^T |\Delta_i(t)|^2 \mathrm{d}t \quad \text{Tone Estimation}$$

$$\gtrsim \frac{1}{T} \int_0^T |\sum_{i=1}^{k} \Delta_i(t)|^2 \mathrm{d}t \quad \text{Signal Estimation}$$

Tone Estimation to Signal Estimation

Define $\Delta_i(t) = a_i(t) - a'_i(t) \;= v_i e^{2\pi \mathbf{i} f_i t} - v'_i e^{2\pi \mathbf{i} f'_i t}$



$$\mathcal{N}^2 \gtrsim \sum_{i=1}^{k} \frac{1}{T} \int_0^T |\Delta_i(t)|^2 \mathrm{d}t \quad \text{Tone Estimation}$$

$$\gtrsim \frac{1}{T} \int_0^T \left| \sum_{i=1}^{k} \Delta_i(t) \right|^2 \mathrm{d}t \quad \text{Signal Estimation}$$

$$= \frac{1}{T} \int_0^T | x_k(t) - x'(t) |^2 \mathrm{d}t$$

Extremely Simplified Proof

$$\text{Goal}: k\sum_{i=1}^{k} y_i^2 \geqslant \left(\sum_{i=1}^{k} y_i\right)^2$$

$$\left(\sum_{i=1}^{k} y_i\right)^2$$

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Define $\Delta_i(t) = a_i(t) - a_i'(t) = v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$

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$$\frac{1}{T} \int_0^T |\sum_{i=1}^{k} \Delta_i(t)|^2 \mathrm{d}t$$

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$$= \sum_{i=1}^{k} \frac{1}{T} \int_0^T |\, \Delta_i(t) \,|^2 \mathrm{d}t + \sum_{i \neq j} \frac{1}{T} \int_0^T \Delta_i(t) \overline{\Delta_j(t)} \, \mathrm{d}t$$

Simplifed Proof

Define $\Delta_i(t) = a_i(t) - a_i'(t) \; = v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$

Goal : $\sum_{i=1}^k \frac{1}{T} \int_0^T |\Delta_i(t)|^2 \mathrm{d}t \gtrsim \frac{1}{T} \int_0^T | \sum_{i=1}^k \Delta_i(t)|^2 \mathrm{d}t$

$$\frac{1}{T} \int_0^T | \sum_{i=1}^k \Delta_i(t)|^2 \mathrm{d}t$$

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$$\lesssim (1 + \frac{\log^2(k)}{T\eta}) \cdot \sum_{i=1}^k \frac{1}{T} \int_0^T | \Delta_i(t) |^2 \mathrm{d}t$$

Simplifed Proof

$$\text{Define } \Delta_i(t) = a_i(t) - a_i'(t) \;= v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$$

$$\text{Goal : } \sum_{i=1}^k \frac{1}{T}\int_0^T |\Delta_i(t)|^2 \mathrm{d}t \gtrsim \frac{1}{T}\int_0^T |\sum_{i=1}^k \Delta_i(t)|^2 \mathrm{d}t$$

$$\frac{1}{T}\int_0^T |\sum_{i=1}^k \Delta_i(t)|^2 \mathrm{d}t$$

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$$\lesssim \sum_{i=1}^k \frac{1}{T}\int_0^T |\,\Delta_i(t)\,|^2 \mathrm{d}t \;\text{ for } T > \log^2(k)/\eta$$

Simplifed Proof

Define $\Delta_i(t) = a_i(t) - a_i'(t) \; = v_i e^{2\pi \mathbf{i} f_i t} - v_i' e^{2\pi \mathbf{i} f_i' t}$

Goal : $\sum_{i=1}^{k} \frac{1}{T} \int_0^T |\Delta_i(t)|^2 \mathrm{d}t \gtrsim \frac{1}{T} \int_0^T | \sum_{i=1}^{k} \Delta_i(t)|^2 \mathrm{d}t$

$$\frac{1}{T} \int_0^T | \sum_{i=1}^{k} \Delta_i(t)|^2 \mathrm{d}t$$

$= \text{diagonal terms} + \text{off-diagonal terms}$

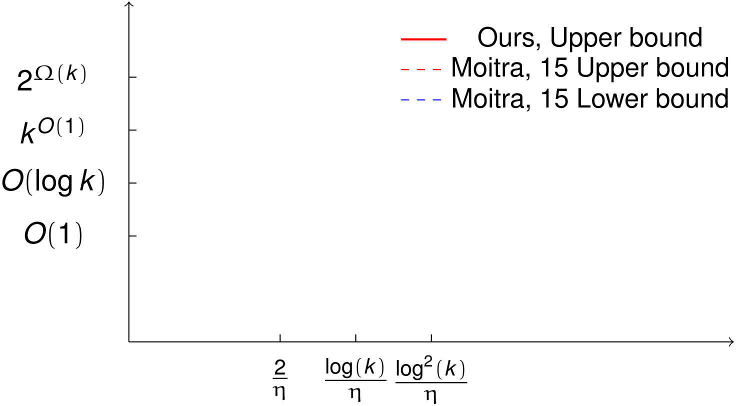
$= \sum_{i=1}^{k} \frac{1}{T} \int_0^T | \Delta_i(t) |^2 \mathrm{d}t + \sum_{i \neq j} \frac{1}{T} \int_0^T \Delta_i(t) \overline{\Delta_j(t)} \, \mathrm{d}t$

$\lesssim \quad \sum_{i=1}^{k} \frac{1}{T} \int_0^T | \Delta_i(t) |^2 \mathrm{d}t \;$ for $T > \log^2(k)/\eta$

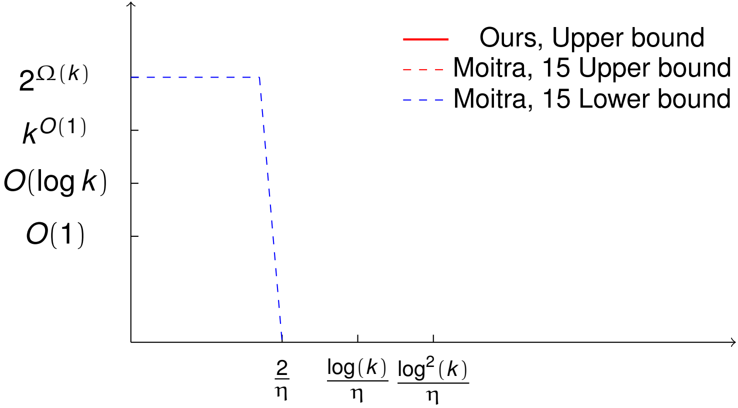
T is large enough, $\Delta_i(t)$ is more likely orthogonal to $\overline{\Delta_j(t)}$, $\forall i \neq j$

Open questions

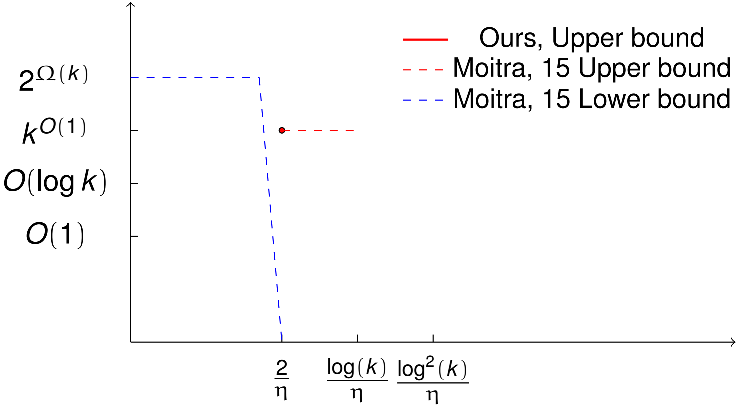
Open questions



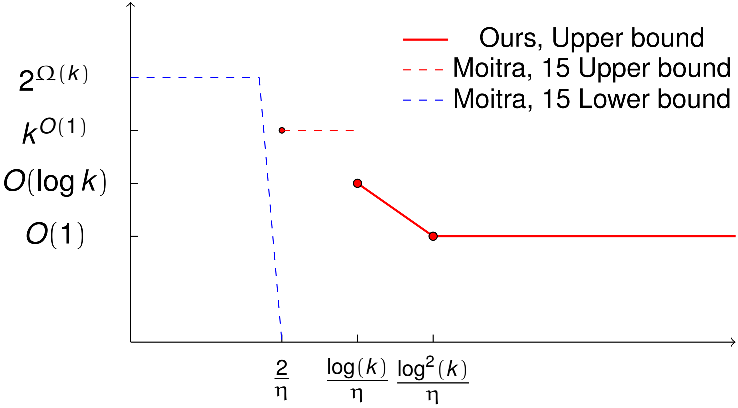
Open questions



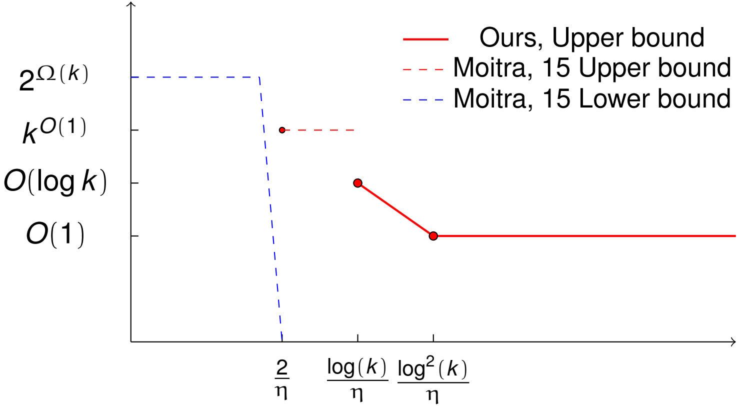
Open questions



Open questions

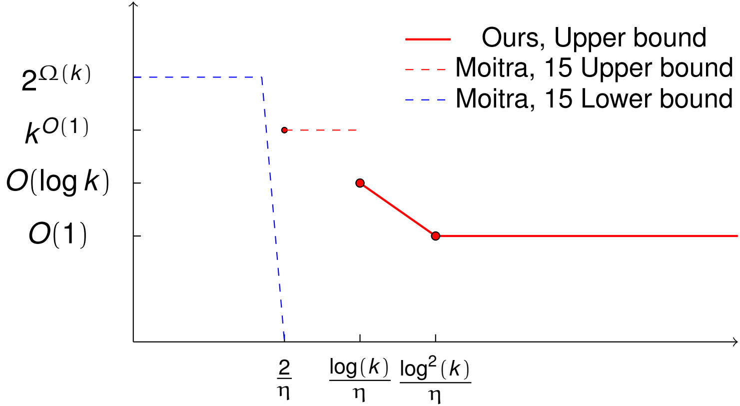


Open questions



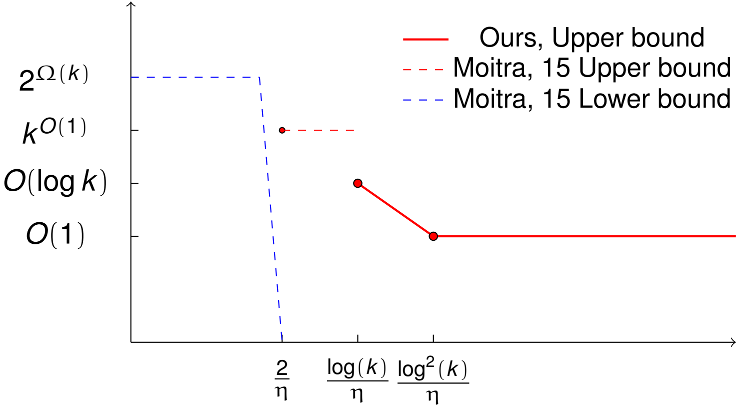
- Can we reconstruct a signal $x'(t)$ without recovering each (v_i, f_i) nicely?

Open questions



- Can we reconstruct a signal $x'(t)$ without recovering each (v_i, f_i) nicely?
- Noise is exponentially small in k, how small duration T can we pick?

Open questions



- Can we reconstruct a signal $x'(t)$ without recovering each (v_i, f_i) nicely?
- Noise is exponentially small in k, how small duration T can we pick?
- Improve our constant approximation result to $(1 \pm \epsilon)$ approximation by increasing the sample duration T?

Summary

- DFT setting: $\log^d \log n$ far from optimal in d dimensions.
- Continuous setting: more to learn.

Thank You