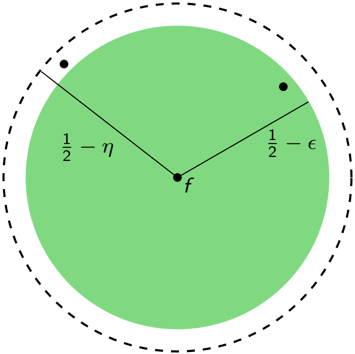
# Algorithmic Questions in Higher-Order Fourier Analysis





Madhur Tulsiani

TTI Chicago


Based on joint works with Arnab Bhattacharyya, Eli Ben-Sasson, Pooya Hatami, Noga Ron-Zewi, Luca Trevisan, Salil Vadhan and Julia Wolf

# Decomposition Theorems

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Object of study

Family of algorithms or functions

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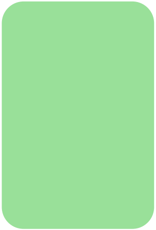

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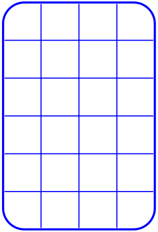

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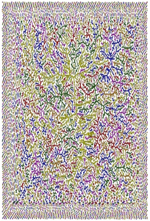


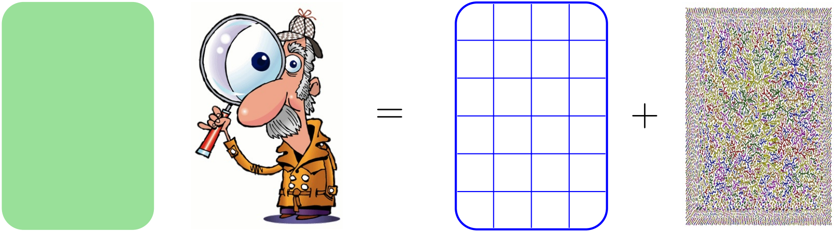
Object of study

Family of algorithms or functions

Structured

No apparent structure (Pseudorandom)

# Decomposition Theorems



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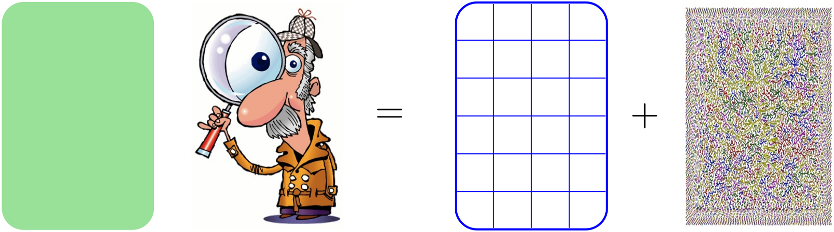

Object of study | Family of algorithms or functions = Structured + No apparent structure (Pseudorandom)

- Decompose an object in to structured and pseudorandom parts.
- Can often ignore the pseudorandom part for many applications. Structured part easier to study.

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- [Parseval]: $\|g\|^2 \;=\; \langle g, g \rangle \;=\; \mathbb{E}_x \left[ (g(x))^2 \right] \;=\; \sum_\alpha \left( \hat{g}(\alpha) \right)^2$.

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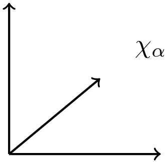

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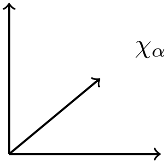

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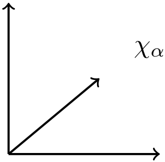

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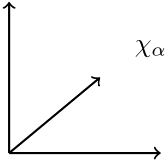

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- $k \leq 1/\epsilon^2$. simple structure
- $f$ has small correlation with linear functions. pseudorandom

$$\forall \alpha, |\langle f, \chi_\alpha \rangle| = |\mathbb{E}_x [f(x)\chi_\alpha(x)]| \leq \epsilon$$

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$$\|f\|_{U^2}^4 = \mathbb{E}_{x,y,z}\left[f(x) \cdot f(x+y) \cdot f(x+z) \cdot f(x+y+z)\right].$$

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Think $f =$ indicator of a set. $\|f\|_{U^2}$ counts 2-dimensional "boxes" in set.

- Can define higher norms similarly

$$\|f\|_{U^3}^8 = \mathbb{E}_{x,y,z,w}\left[\begin{array}{l} \mathrm{f(x)\ f(x{+}y)\ f(x{+}z)\ f(x{+}y{+}z)} \\ \mathrm{f(x{+}w)\ f(x{+}y{+}w)\ f(x{+}z{+}w)\ f(x{+}y{+}z{+}w)} \end{array}\right]$$

# Norms, Shnorms... so what?

- $\|f\|_{U^2}$ measures correlation with Fourier characters (linear phase functions).

$$\left(\max_{\alpha}\left|\hat{f}(\alpha)\right|\right)^4 \le \|f\|_{U_2}^4 \le \left(\max_{\alpha}\left|\hat{f}(\alpha)\right|\right)^2$$

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  - $\|f\|_{U^3} \ge \epsilon \implies$ for some $Q, \left|\langle f, (-1)^Q \rangle\right| \ge \eta(\epsilon)$.

# Decompositions in Quadratic Fourier Analysis

**Theorem (Gowers-Wolf 09)**

*Given* $\epsilon > 0$*, any* $g : \mathbb{F}_2^n \to [-1,1]$ *can be decomposed as*

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Similar to basic Fourier decomposition, where we get

$$g = \sum_{i=1}^{k} c_i \chi_{\alpha_i}(x) + f,$$

with $|\langle f, \chi_\alpha \rangle| \leq \epsilon$ for all $\alpha$ and $k \leq 1/\epsilon^2$ (also implies $\sum_i |c_i| \leq 1/\epsilon$).

# Decompositions in Higher-Order Fourier Analysis

**Theorem (Gowers-Wolf 10)**

*Given $\epsilon > 0$ and $p > d$, there exists $M(\epsilon, p)$ such that any $g : \mathbb{F}_p^n \to [-1, 1]$ can be decomposed as*

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- Stronger decomposition theorems proved by [HL 11] and [BFL 12].
- Decomposition theorems for the case when $p \leq d$ require non-classical polynomials.

Q1: Can we compute these decompositions efficiently?

# Algorithmic version of the basic Fourier decomposition

**Theorem (Goldreich-Levin 89)**

*There is a randomized algorithm, which given* $\epsilon, \delta > 0$ *and oracle access to* $g : \mathbb{F}_2^n \to [-1, 1]$, *runs in time* $O\left(n^2 \log n \cdot (1/\epsilon^2) \cdot \log(1/\delta)\right)$ *and outputs a decomposition*

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- Finding large Fourier coefficients has many applications.

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$$\sum c_i(-1)^{Q_i} + f$$

$$s.t. \sum_i |c_i| \leq M(\epsilon), \|f\|_{U^3} \leq \epsilon$$

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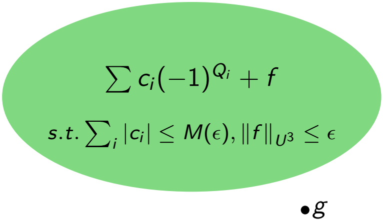

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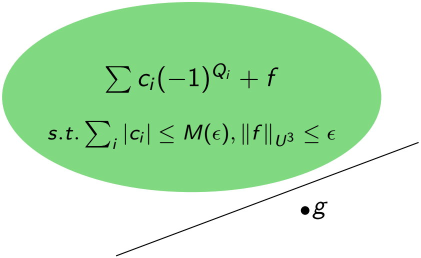

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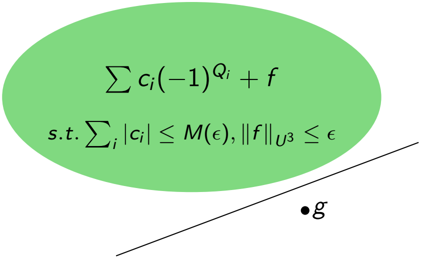


- Use inverse theorem for Gowers norm to get a contradiction.

# A quadratic Goldreich-Levin Theorem

**Theorem (T, Wolf 11)**

*For* $M(\epsilon) = \exp(1/\epsilon^C)$, *can compute in time poly*$(n, M(\epsilon), \log(1/\delta))$, *a decomposition*

$$g = \sum_{i=1}^{k} c_i (-1)^{Q_i} + f + e$$

*such that*

- *with probability* $1 - \delta$, $\|f\|_{U^3} \leq \epsilon$ *and* $\|e\|_1 \leq \epsilon$.
- $\sum_i |c_i| \leq M(\epsilon)$ *and* $k \leq (M(\epsilon))^2$.

# Improved quadratic Goldreich-Levin Theorem

**Theorem (BRTW 12)**

*For* $M(\epsilon) = O(\exp(\log^4(1/\epsilon)))$, *can compute in time* $poly(n, M(\epsilon), \log(1/\delta))$, *a decomposition*

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Goal: Given $g : \mathbb{F}_2^n \to [-1, 1]$, find a decomposition $g = \sum_i c_i(-1)^{Q_i} + f$ such that $\|f\|_{U^3} \leq \epsilon$.

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Convergence: $\|f_{t-1}\|^2 - \|f_t\|^2 = 2\eta \langle f_{t-1}, (-1)^{Q_t} \rangle - \eta^2 \ge \eta^2$.

[Samorodnitsky 07]: $\forall Q \left|\langle (-1)^Q, f \rangle\right| \le \eta(\epsilon) \implies \|f\|_{U^3} \le \epsilon$.

# A constructive proof of decomposition [TTV 09]

Goal: Given $g : \mathbb{F}_2^n \to [-1,1]$, find a decomposition $g = \sum_i c_i(-1)^{Q_i} + f$ such that $\|f\|_{U^3} \le \epsilon$.

Algorithm:

- $h_0 = 0$, $f_0 = g - h_0$, $t = 1$.
- while there is a quadratic function $Q_t$ such that $\langle f_{t-1}, (-1)^{Q_t} \rangle > \eta$
  - $h_t \;=\; h_{t-1} + \eta \cdot (-1)^{Q_t} \;=\; \sum_{r=1}^{t} \eta \cdot (-1)^{Q_r}$
  - $f_t \;=\; g - h_t$
  - $t \;=\; t + 1$
- return $h_t$

Convergence: $\|f_{t-1}\|^2 - \|f_t\|^2 \;=\; 2\eta \langle f_{t-1}, (-1)^{Q_t} \rangle - \eta^2 \;\ge\; \eta^2$.

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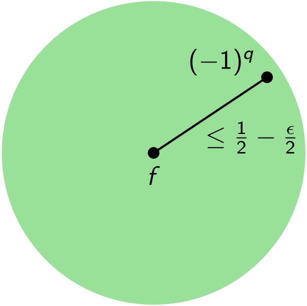
Truth-tables of functions $(-1)^Q$ form the Reed-Muller code of order 2. Want a codeword inside a ball of distance $1/2 - \epsilon/2$ around $f$ (if one exists).



Q2: Decoding beyond the list-decoding radius

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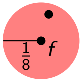

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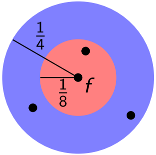


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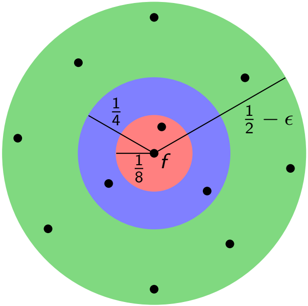


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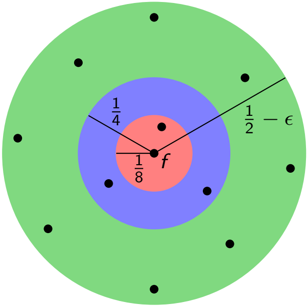


- List decoding radius is $\frac{1}{4}$.
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- But we only need to find one codeword! In time $poly(n)$ (polylogarithmic in code length).

# Finding codewords at large distances

- Given (the coefficients of) a degree-$d$ polynomial $P : \mathbb{F}_p^n \to \mathbb{F}_p$, the Reed-Muller encoding of $P$ is of length $p^n$ and is given by the table of values $\{P(x)\}_{x \in \mathbb{F}_p^n}$.

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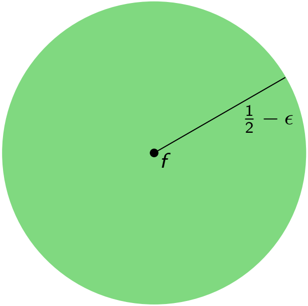
- If there exists a Reed-Muller codeword within a ball of radius $1 - \frac{1}{p} - \epsilon$, find one within a ball of radius $1 - \frac{1}{p} - \eta$.

# Finding a single codeword: the quadratic case



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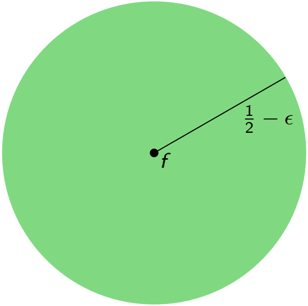


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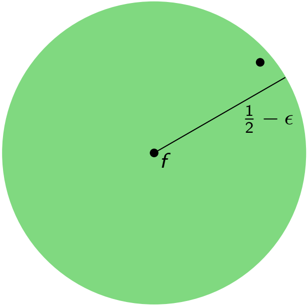


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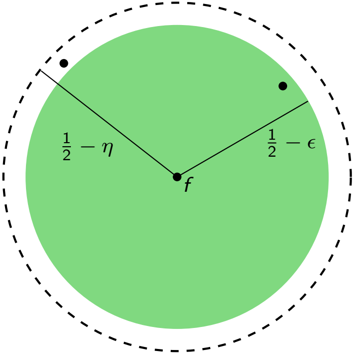


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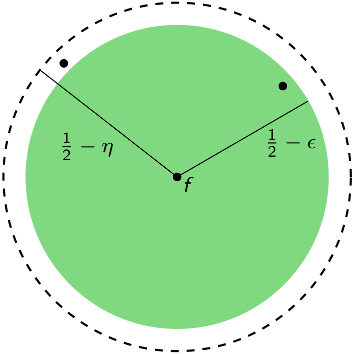


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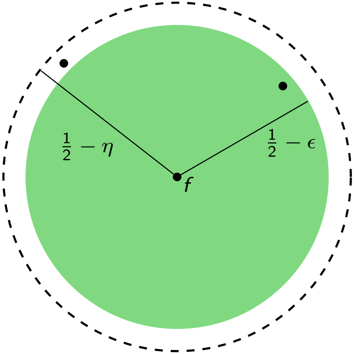


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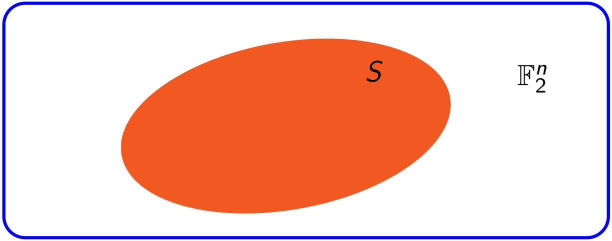
- First example of any kind of decoding beyond the list decoding radius.

# Algorithmic versions of combinatorial theorems

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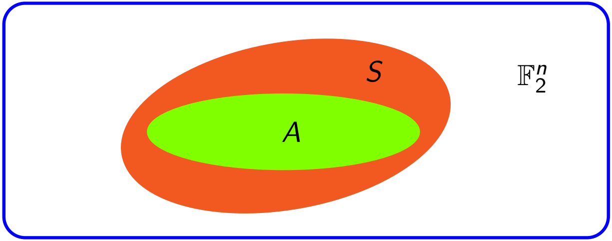


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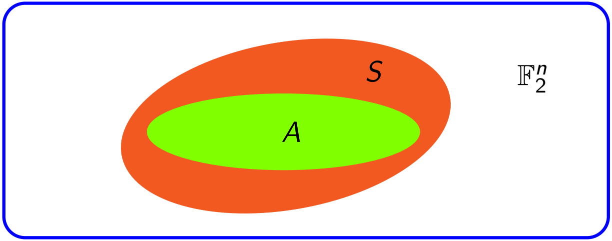


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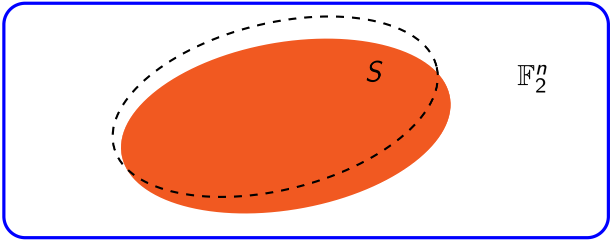


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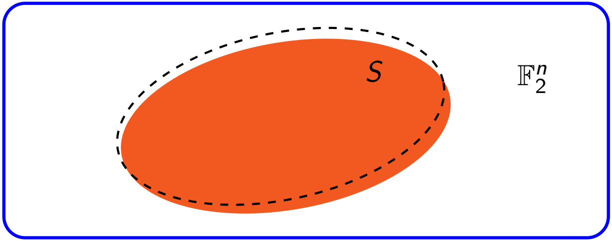


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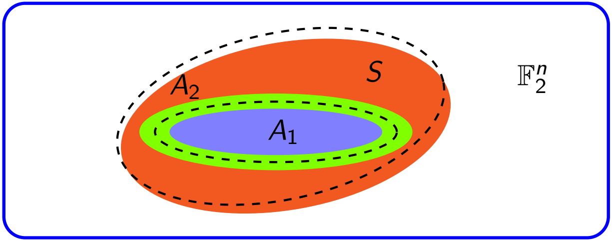


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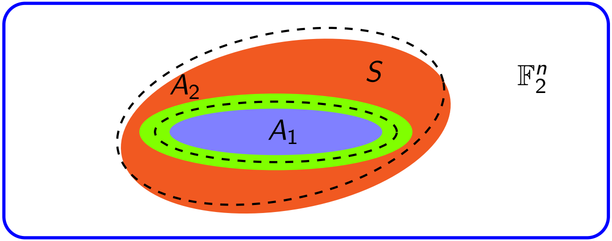


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- Prove "robust" versions of theorems from additive combinatorics.

# Finding subspace structure

Most combinatorial results used here find and refine subspace structure in $S \subseteq \mathbb{F}_2^n$.

- [BSG]: If $\mathbb{P}_{x,y\in S}[x+y\in S] \geq \epsilon$ then $\exists A \subseteq S$ s.t.

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- [CS 09]: If $|A + A| \leq K \cdot |A|$, then $\mathbf{1}_A * \mathbf{1}_A$ has a large set of "almost periods" i.e., there is a large set $X \subseteq \mathbb{F}_2^n$ s.t

$$\mathbf{1}_A * \mathbf{1}_A(\cdot) \approx \mathbf{1}_A * \mathbf{1}_A(\cdot + x) \quad \forall x \in X$$

$\mathbf{1}_A * \mathbf{1}_A(\cdot) \approx$ distribution of sum of two random elements from $A$.

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- [BRTW 14]: Sampling-based proof of [CS 09]. Improved quadratic Goldreich-Levin.
- Question: Can sampling based proofs be used to find better subspace structure?

# Decompositions for higher-degrees

- Question: Given $F : \mathbb{F}_p^n \to \mathbb{F}_p$, does there exist a polynomial $P \in \mathcal{P}_d$ such that $\left|\left\langle \omega^F, \omega^P \right\rangle\right| \geq \epsilon$? If yes, find one.

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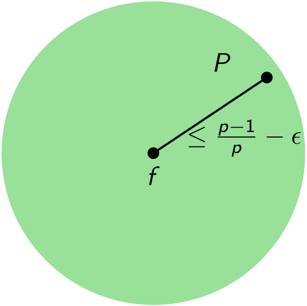

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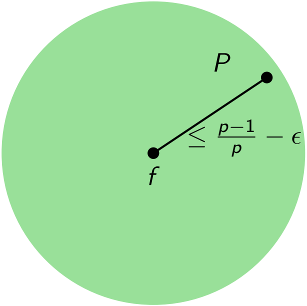


- Can be solved for the special case when $F \in \mathcal{P}_k$ and $p > k$, inverse theorem by [GT 09].

# Decomposition Theorems and Regularity

- [GT 09]: Actually prove a decomposition theorem when $F \in \mathcal{P}_k$:

$$\omega^F = \Gamma(P_1, \ldots, P_m) + f_2$$

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- Here, $\Gamma : \mathbb{F}_p^m \to \mathbb{R}$. By (linear) Fourier analysis

$$\Gamma(P_1, \ldots, P_m) \;=\; \sum_{c_1, \ldots, c_m} \widehat{\Gamma}(c_1, \ldots, c_m) \cdot \omega^{\sum_i c_i P_i}$$

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- Proof by [GT 09] and many other applications require the factor $\mathcal{B} = \{P_1, \ldots, P_m\}$ to satisfy certain “regularity” properties. Obtaining regularity is the main challenge in converting their proof to an algorithm.

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- Like Szemerédi's regularity lemma, proofs find a certificate of non-regularity and make progress by local modification.

# Q3: Algorithmic Regularity Lemmas

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- Show these notions provide required equidistribution for various known applications.

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- Do algorithms really need to be derived from proofs of existence? Can there be a simpler algorithm for which a solution is guaranteed by the proof?
- Regularity lemmas give terrible quantitative bounds. Is there a way to use weaker regularity properties and obtain better bounds?

Thank You

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Questions?