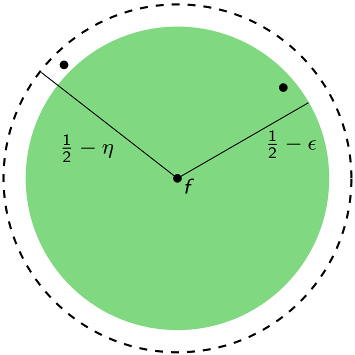
Algorithmic Questions in Higher-Order Fourier Analysis



Madhur Tulsiani

TTI Chicago

Based on joint works with Arnab Bhattacharyya, Eli Ben-Sasson, Pooya Hatami, Noga Ron-Zewi and Julia Wolf

Decomposition Theorems

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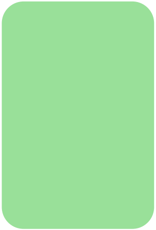
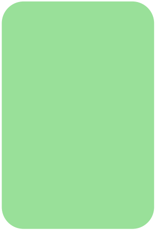


Object of study

Family of algorithms or functions

Decomposition Theorems



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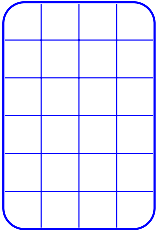
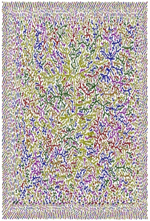


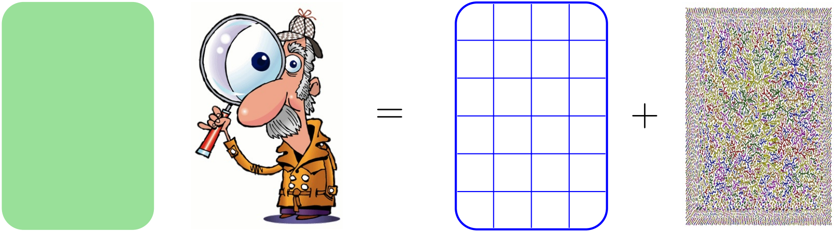
Structured

+



No apparent structure (Pseudorandom)

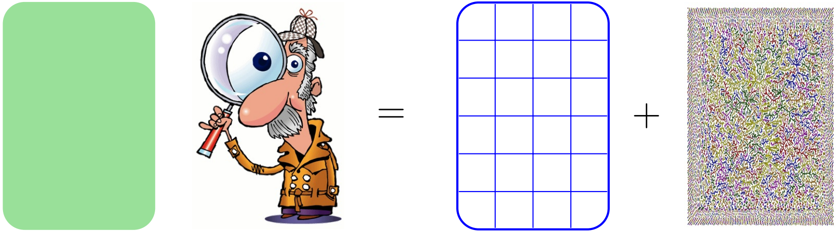
Decomposition Theorems



Object of study | Family of algorithms or functions = Structured + No apparent structure (Pseudorandom)

- Decompose an object in to structured and pseudorandom parts.

Decomposition Theorems



Object of study | Family of algorithms or functions | Structured | No apparent structure (Pseudorandom)

- Decompose an object in to structured and pseudorandom parts.
- Can often ignore the pseudorandom part for many applications. Structured part easier to study.

A basic decomposition in Fourier analysis

$$g : \mathbb{F}_2^n \to [-1, 1]$$

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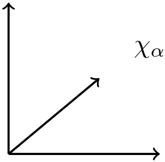

$$\chi_\alpha(x) = (-1)^{\alpha \cdot x} = (-1)^{\sum_i \alpha_j x_j}$$

$$\alpha \in \mathbb{F}_2^n$$

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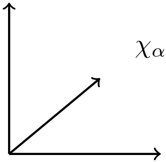

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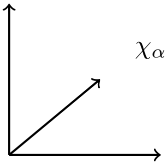

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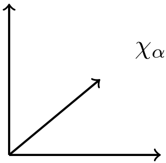

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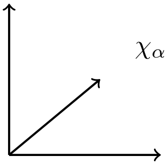

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- f has small correlation with linear functions. For any α,
$$|\langle f, \chi_\alpha \rangle| = |\mathbb{E}_x [f(x)\chi_\alpha(x)]| \leq \epsilon$$
- f is pseudorandom and can be ignored in many applications of Fourier analysis.

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$$\|f\|_{U^3}^8 = \mathbb{E}_{x,y,z,w}\left[\begin{array}{l} f(x)\ f(x+y)\ f(x+z)\ f(x+y+z) \\ f(x+w)\ f(x+y+w)\ f(x+z+w)\ f(x+y+z+w) \end{array}\right]$$

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  - $\|f\|_{U^3} \geq \epsilon \implies$ for some Q, $\left|\langle f, (-1)^Q \rangle\right| \geq \eta(\epsilon)$.

Decompositions in Quadratic Fourier Analysis

Theorem (Gowers-Wolf 09)

Given $\epsilon > 0$, *any* $g : \mathbb{F}_2^n \to [-1, 1]$ *can be decomposed as*

$$g \;=\; \sum_{i=1}^{k} c_i (-1)^{Q_i} + f + e$$

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Similar to basic Fourier decomposition, where we get

$$g = \sum_{i=1}^{k} c_i \chi_{\alpha_i}(x) + f,$$

with $|\langle f, \chi_\alpha \rangle| \leq \epsilon$ for all α and $k \leq 1/\epsilon^2$ (also implies $\sum_i |c_i| \leq 1/\epsilon$).

Decompositions in Higher-Order Fourier Analysis

Theorem (Gowers-Wolf 10)

Given $\epsilon > 0$ *and* $p > d$*, there exists* $M(\epsilon, p)$ *such that any* $g : \mathbb{F}_p^n \to [-1, 1]$ *can be decomposed as*

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for $P_1, \ldots, P_k \in \mathcal{P}_d$ *(polynomials of degree at most* d*) such that*

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- Stronger decomposition theorems proved by [HL 11] and [BFL 12].
- Decomposition theorems for the case when $p \leq d$ require non-classical polynomials.

Q1: Can we compute these decompositions efficiently?

Algorithmic version of the basic Fourier decomposition

Theorem (Goldreich-Levin 89)

There is a randomized algorithm, which given $\epsilon, \delta > 0$ *and oracle access to* $g : \mathbb{F}_2^n \to [-1, 1]$, *runs in time* $O\left(n^2 \log n \cdot (1/\epsilon^2) \cdot \log(1/\delta)\right)$ *and outputs a decomposition*

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- Finding large Fourier coefficients has many applications.

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$$\sum c_i (-1)^{Q_i} + f$$

$$s.t. \sum_i |c_i| \leq M(\epsilon), \|f\|_{U^3} \leq \epsilon$$

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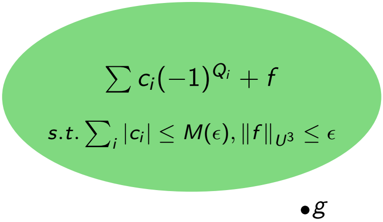

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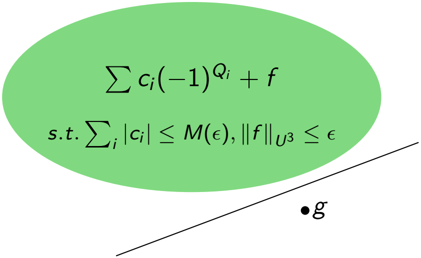

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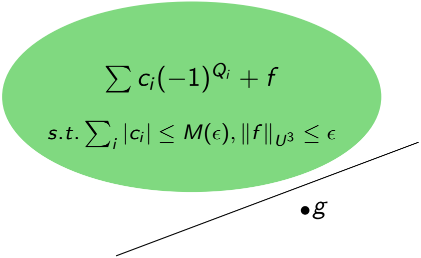

- Use inverse theorem for Gowers norm to get a contradiction.

A quadratic Goldreich-Levin Theorem

Theorem (T, Wolf 11)

For $M(\epsilon) = \exp(1/\epsilon^C)$*, can compute in time* $poly(n, M(\epsilon), \log(1/\delta))$*, a decomposition*

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such that

- *with probability* $1 - \delta$, $\|f\|_{U^3} \le \epsilon$ *and* $\|e\|_1 \le \epsilon$.
- $\sum_i |c_i| \le M(\epsilon)$ *and* $k \le (M(\epsilon))^2$.

Improved quadratic Goldreich-Levin Theorem

Theorem (BRTW 12)

For $M(\epsilon) = O(\exp(\log^4(1/\epsilon)))$, *can compute in time* $poly(n, M(\epsilon), \log(1/\delta))$, *a decomposition*

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The algorithmic problem

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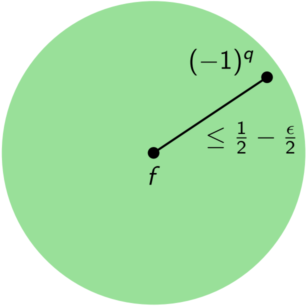
Truth-tables of functions $(-1)^Q$ form the Reed-Muller code of order 2. Want a codeword inside a ball of distance $1/2 - \epsilon/2$ around f (if one exists).



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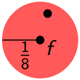

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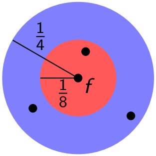

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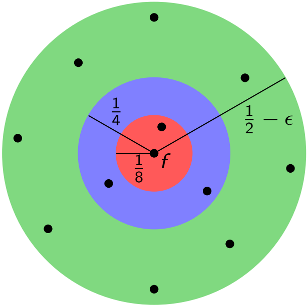

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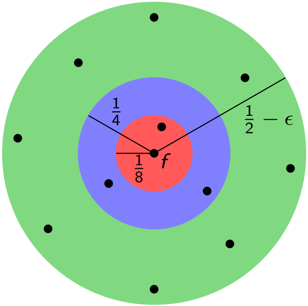

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- But we only need to find one codeword! In time *poly*(n) (polylogarithmic in code length).

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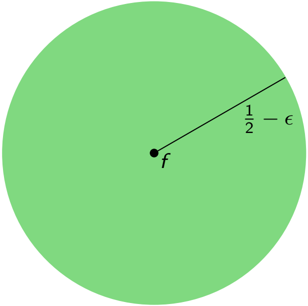

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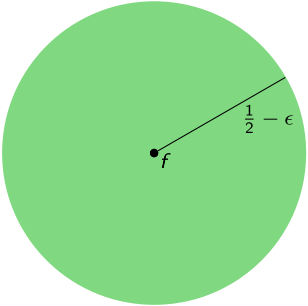

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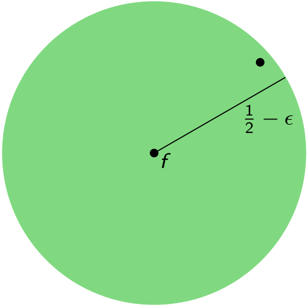

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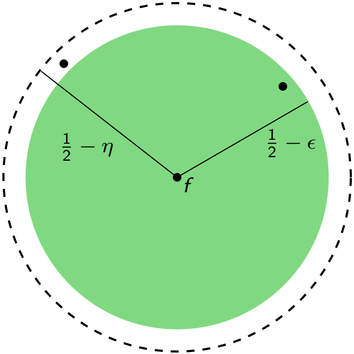

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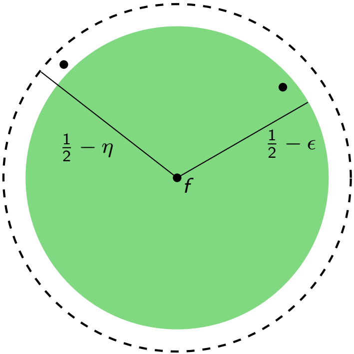

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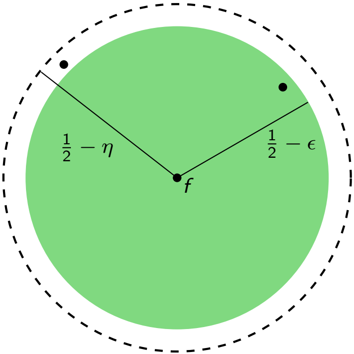

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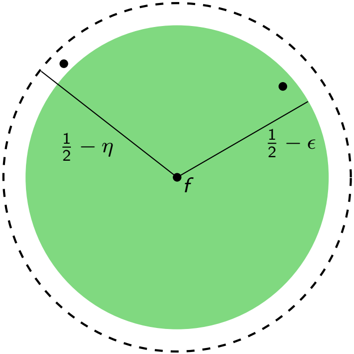

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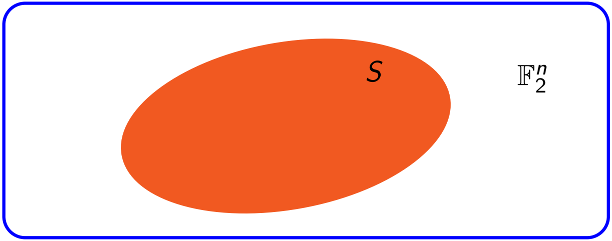
- First example of any kind of decoding beyond the list decoding radius.

Algorithmic versions of combinatorial theorems



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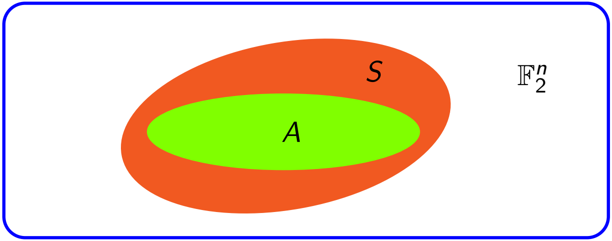

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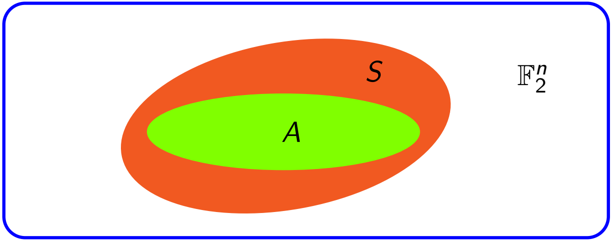

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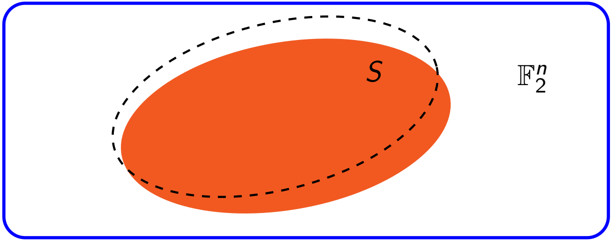

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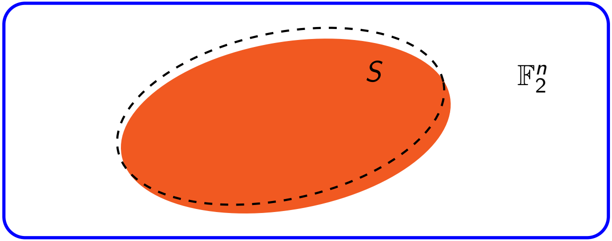

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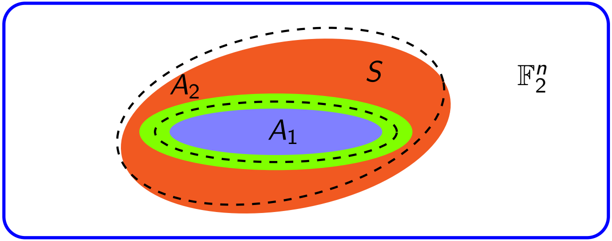

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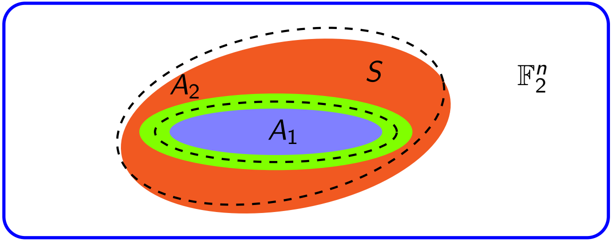

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- May be useful for other applications.

Finding subspace structure

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- [BRTW 14]: Sampling-based proof of [CS 09]. Improved quadratic Goldreich-Levin.

Decompositions for higher-degrees

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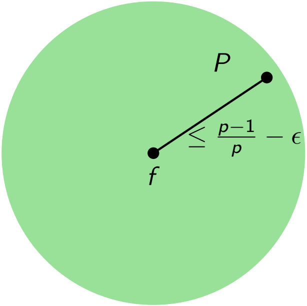

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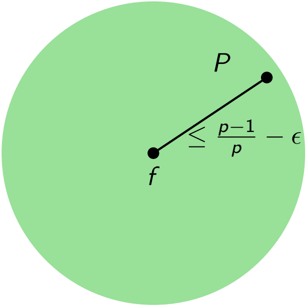

- Can be solved for the special case when $F \in \mathcal{P}_k$ and $p > k$, inverse theorem by [GT 09].

Decomposition Theorems and Regularity

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$$\omega^F = \Gamma(P_1, \ldots, P_m) + f_2$$

where $P_1, \ldots, P_m \in \mathcal{P}_d$ and $\|f_2\|_{U^{d+1}} \leq \epsilon$.

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- Proof by [GT 09] and many other applications require the factor $\mathcal{B} = \{P_1, \ldots, P_m\}$ to satisfy certain "regularity" properties. Obtaining regularity is the main challenge in converting their proof to an algorithm.

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- Like Szemerédi's regularity lemma, proofs find a certificate of non-regularity and make progress by local modification.

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- Show these notions provide required equidistribution for various known applications.

Further questions

- Higher-degree decomposition theorems.
- (Approximate) Decoding beyond the list decoding radius for other codes. Even for distances slightly beyond the list-decoding radius.
- Do algorithms really need to be derived from proofs of existence? Can there be a simpler algorithm for which a solution is guaranteed by the proof?
- Applications of algorithmic decomposition theorems.

Thank You

Questions?