Regularity for Polynomials and Linear Forms

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(joint work with Hamed Hatami and Shachar Lovett)

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Setting

- $f : \mathbb{F}^n \to R$
  - $\mathbb{F} = \mathbb{F}_p$.
  - p is a fixed prime, and n is large
- $e(x) := e_p(x) := e^{2\pi i x/p}$.
- $x, y, z \in \mathbb{F}^n$, $X, Y, Z \in (\mathbb{F}^n)^k$.

Fourier Analysis and Higher-order Fourier Analysis

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Applications

Useful in controling several expressions regarding a given function, such as approximate Linearity, density of 3-term APs.

Approximate Linearity (As seen in an analysis of BLR test)

$f : \mathbb{F}_2^n \to \{0, 1\}$, letting $g(x) = (-1)^{f(x)}$

$$\Pr_{x,y}(f(x+y) = f(x) + f(y)) = \frac{1}{2} + \frac{1}{2}\mathbb{E}_{x,y}(g(x+y)g(x)g(y))$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{\sigma_1,\sigma_2,\sigma_3 \in \mathbb{F}_2^n} \widehat{g}_{\sigma_1}\widehat{g}_{\sigma_2}\widehat{g}_{\sigma_3}\mathbb{E}_{x,y} e_2(\sigma_1^t x + \sigma_2^t y + \sigma_3^t (x+y))$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{\sigma \in \mathbb{F}_2^n} \widehat{g}_\sigma^3 \leq \max_\sigma \widehat{g}_\sigma$$

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Correlation with characters captures approximate linearity.

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Correlation with characters can control density of 3-term APs.

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- More complex behavior, such as 4-APs.

- Need approximation of functions by a linear combination of these higher-order polynomials.

Decomposition Theorems as a result of Inverse Theorems

[Bergelson, Green, Samorodnitsky, Szegedy, Tao, Ziegler]

$$f \approx_{U^{d+1}} \Gamma(P_1, ..., P_C),$$

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No Orthogonality, unlike in classical Fourier analysis!

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Can assume $P_1, ..., P_C$ in $f \approx \Gamma(P_1, ..., P_C)$ is a high-rank collection.

Figure: f

Figure: Approximation by polynomials: $\Gamma(P_1, ..., P_C)$

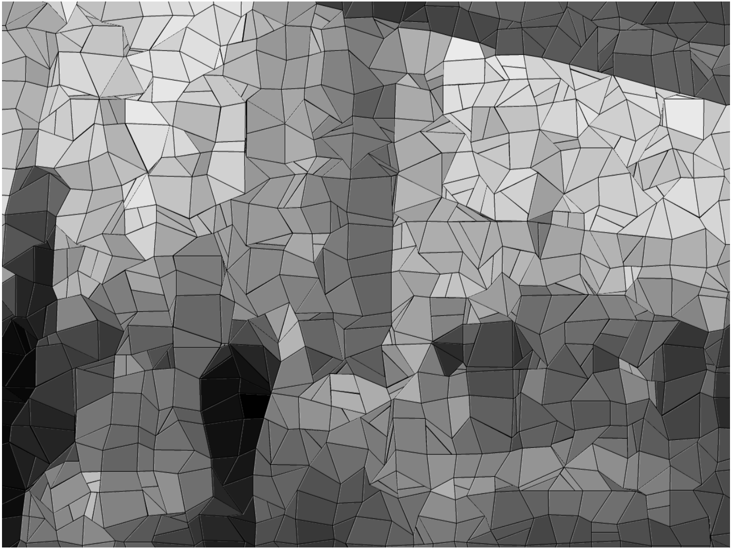

Figure: Regular refinement: $\Gamma'(Q_1, ..., Q_c)$

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Developed in order to understand more complex averages.

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We need stronger near-orthogonality over sets of linear forms!

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- Need to analyze the distribution of $f|_V$.
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- $f \approx \Gamma(P_1(x), \ldots, P_C(x))$.

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- Let $L_1, \ldots, L_{p^k}$ be the points of a random V.
- $f \approx \Gamma(P_1(x), \ldots, P_C(x))$.
- We need to understand the joint distribution $\left(P_i(L_j(X))\right)_{i \in [C], j \in [p^k]}$.

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- Cannot expect almost independence for all entries!
  - e.g. $\deg(P) = 1$, then $P(x+y) + P(z) = P(x) + P(y+z)$.
  - e.g. $\deg(P) < d$, then $\sum_{\omega \in \{0,1\}^{d+1}} (-1)^{|\omega|} P(X + \sum_{i \in \omega} Y_i) = 0$.

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- General case: [H. Hatami, P.H., and Lovett, General systems of linear forms].

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Columns are almost independently distributed.

Theorem. (Near Orthogonality [Hamed Hatami, P.H., Lovett])

$P_1, \ldots, P_C$ be a high-rank set of polynomials. Let

$$P_\Lambda(X) = \sum_{i \in [C], j \in [m]} \lambda_{i,j} P_i(L_j(X)).$$

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$P_\Lambda \equiv 0$ if and only if the same is true for any collection of same degree polynomials.

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Proof Ideas.

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- Reduce to the case when $|L_j| \leq \deg(P_i)$.
  - e.g. $Q(2x+z) = 2Q(x) + Q(z) - 2Q(x+z) - Q(2x)$.

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- Applications of certain derivative operators D_i s.t.

$$(|\mathbb{E}_X e(P_\Lambda(X))|)^{2^d} \leq \mathbb{E}\left[e((D_1 \cdots D_d P_\Lambda)(X))\right] = \|e(\sum_{i\in C} \lambda'_i P_i)\|_{U^d}^{2^d}$$

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[H.Hatami, P.H., Lovett]:

- Define a notion of homogeneity for nonclassical polynomials, $P(cx) = \lambda_c P(x)$.
- Show that every degree-d polynomial can be written as linear combination of homogeneous polynomials.

“Application”

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$$\|f\|_{U^{s'}} \leq \delta(\epsilon) \Rightarrow |\mathbb{E} f(M_1(X)) \cdots f(M_\ell(X))| \leq \epsilon \quad \text{with } s' < s+1.$$

True Complexity [Gowers-Wolf]

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- [H. Hatami-Lovett] When $\mathrm{CS} - \mathrm{Complexity} < |\mathbb{F}|$.
- [H. Hatami-P.H.-Lovett] Verify the conjecture in its full generality.

A simple telescoping (hybrid) argument leads to:

Corollary.

Assume that $L_1^{d+1}, \ldots, L_m^{d+1}$ are linearly independent. Then $\|f - g\|_{U^{d+1}} \leq \delta(\epsilon)$ implies

$$\left| \mathbb{E}_X \left[\prod_{i=1}^{m} f(L_i(X)) \right] - \mathbb{E}_X \left[\prod_{i=1}^{m} g(L_i(X)) \right] \right| \leq \epsilon$$

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$\|1_A - 1_B\|_{U^{d+1}} \leq \delta$ implies that the number of d-APs in A and B are similar.

True Complexity

Theorem (H. Hatami-P.H.-Lovett)

Let $L_1, ..., L_m$ be such that L_1^{d+1} is not in the span of $L_2^{d+1}, ..., L_m^{d+1}$. Then

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Proof steps.

- We may assume that d is less than the CS-Complexity.

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$$\mathbb{E}\left[(g_i + h_i)(L_1(X)) \cdots (g_m + h_m)(L_m(X))\right] \approx \mathbb{E}\left[g_i(L_1(X)) \cdots g_m(L_m(X))\right]$$

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Thanks!